In general, a geometric series will have this form:

$$s = a + aR + aR^2 + aR^3 + aR^4 + \dots$$

$\times R \quad \times R \quad \times R \quad \times R$ → common ratio

$$\boxed{s = \sum_{n=1}^{\infty} aR^{n-1}}$$ → form of any geometric series.

Key result on geometric series:

$\sum_{n=1}^{\infty} aR^{n-1}$
- $-1 < R < 1$ → series converges
- $R \geq 1$ or $R \leq -1$ → series diverges.

If converges, $$\boxed{\sum_{n=1}^{\infty} aR^{n-1} = \frac{a}{1-R}}$$

If the absolute value of the common ratio of a geometric series is less than one, it will converge and it converges to the quantity: $\dfrac{\text{first term}}{1 - \text{common ratio}}$

E.g. $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Common ratio: $\frac{1}{2}$; $\left|\frac{1}{2}\right| < 1$

$\longrightarrow$ converges and it converges to: $\dfrac{\frac{1}{2}}{1-\frac{1}{2}} = \boxed{1}$

E.g. $s = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

common ratio: $-\frac{2}{3}$; $\left|-\frac{2}{3}\right| < 1$

$\longrightarrow$ converges and it converges: $\dfrac{5}{1-\left(-\frac{2}{3}\right)} = 5.$

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Ex. $\sum_{i=1}^{\infty} \frac{2-5^i}{10^i}$.

Q: Does this converge? If it does, find the sum.

(Hint: $\sum_{i=1}^{\infty} \frac{2-5^i}{10^i} = \sum_{i=1}^{\infty} \frac{2}{10^i} - \sum_{i=1}^{\infty} \frac{5^i}{10^i}$)

$$= \sum_{i=1}^{\infty} \frac{2}{10^i} - \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= \left(\frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \cdots\right) - \left(\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right)$$

common ratio $= \frac{1}{10}$

common ratio $= \frac{1}{2}$

$$= \frac{\frac{2}{10}}{1 - \frac{1}{10}} - \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{2}{9} - 1 = \boxed{-\frac{7}{9}}$$

E.g. $\displaystyle\sum_{n=3}^{\infty} \frac{(\ln(x))^{n-3}}{6^n}$

Q: For what value(s) of x the series converges?

$$\sum_{n=3}^{\infty} \frac{(\ln(x))^{n-3}}{6^n} = \frac{1}{6^3} + \frac{\ln(x)}{6^4} + \frac{(\ln(x))^2}{6^5} + \frac{(\ln(x))^3}{6^6} + \cdots$$

$$\times \boxed{\frac{\ln(x)}{6}} \qquad \times \frac{\ln(x)}{6} \qquad \times \frac{\ln(x)}{6}$$

This series is a geometric series with common ratio equals to $\boxed{\dfrac{\ln(x)}{6}}$. For the series to converge, we must have $-1 <$ common ratio < 1.

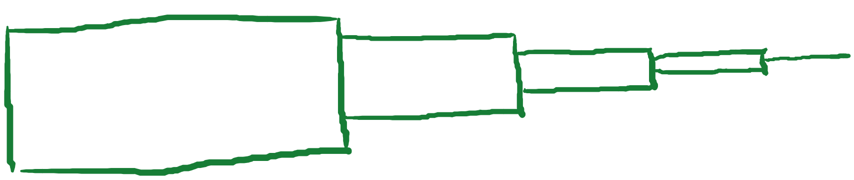
Hence, $-1 < \frac{\ln(x)}{6} < 1 \longrightarrow -6 < \ln(x) < 6$

$\longrightarrow \boxed{e^{-6} < x < e^{6}}$ $0.0025 < x < 403.43$

Q: If it converges, where does it converge to:

$$\text{sum} = \frac{\text{first term}}{1 - \text{common ratio}} = \boxed{\frac{\frac{1}{6^3}}{1 - \frac{\ln(x)}{6}}}$$

* Telescoping Series :



E.g. $\displaystyle\sum_{i=1}^{\infty} \boxed{\frac{1}{(i+1)(i+2)}}$

Apply partial fraction decomposition to term $\dfrac{1}{(i+1)(i+2)}$

$$\frac{1}{(i+1)(i+2)} = \frac{A}{i+1} + \frac{B}{i+2}$$

$\longrightarrow$ Solve for A and B $\longrightarrow$ $A = 1$; $B = -1$

$$\frac{1}{(i+1)(i+2)} = \frac{1}{i+1} - \frac{1}{i+2}$$

$$\sum_{i=1}^{\infty}\left[\frac{1}{i+1} - \frac{1}{i+2}\right] = \underbrace{\left(\frac{1}{2} - \cancel{\frac{1}{3}}\right)}_{i=1} + \underbrace{\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right)}_{i=2}$$

$$+ \underbrace{\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right)}_{i=3} + \underbrace{\left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right)}_{i=4} + \cdots$$

To find the sum properly, we need to analyze the partial sums:

$$\sum_{i=1}^{\infty}\left[\frac{1}{i+1}-\frac{1}{i+2}\right]=\lim_{n\to\infty} S_n$$

where $S_n = \sum_{i=1}^{n}\left[\frac{1}{i+1}-\frac{1}{i+2}\right]$ is the n^{th} partial sum

$$S_n = \underbrace{\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)}_{i=1}+\underbrace{\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)}_{i=2}+\cdots\underbrace{\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)}_{i=n}$$

$$S_n = \frac{1}{2}-\frac{1}{n+2}$$

So, sum of series $= \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)=\boxed{\frac{1}{2}}$