5.2. Series

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An infinite series (or just series) is the sum of the terms of an infinite sequence.

$$\{a_1, a_2, a_3, \dots, a_i, \dots\} \rightarrow \text{sequence } \{a_i\}_{i=1}^{\infty}$$

$$\underbrace{a_1 + a_2 + a_3 + \dots + a_i + \dots}_{} \longrightarrow \text{series}$$

<u>Notation:</u> $\displaystyle\sum_{i=1}^{\infty} a_i$

<u>E.g.</u> $\pi = 3.14159265\dots$

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \dots$$

E.g. $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots\right\}$

We can write this as $\left\{\frac{1}{2^i}\right\}_{i=1}^{\infty}$ (Sequence)

→ Form a series : $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

Write this as : $\sum_{i=1}^{\infty} \frac{1}{2^i}$ (Series)

How do we add up infinitely many terms ?

Add first 2 terms : $\frac{1}{2} + \frac{1}{4} = 0.75$ → 2^{nd} partial sum of the series

______ 3 terms : $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$ → 3^{rd} partial sum

⋮

______ 15 terms : $\frac{1}{2} + \dots + \frac{1}{2^{15}} = 0.9999694$

→ 15^{th} partial sum

$\underline{\text{Idea}}$: $\boxed{\text{Sum of an infinite series} = \text{limit of the sequence of partial sums.}}$

Precise way to define the sum of an infinite series:

$$S = \sum_{i=1}^{\infty} a_i .$$

Consider the sequence of partial sums :

$$S_1 = a_1 ; \quad S_2 = a_1 + a_2 ; \quad S_3 = a_1 + a_2 + a_3 ; \ \dots$$

$$S_n = a_1 + a_2 + \dots + a_n = \sum_{i=1}^{n} a_i \rightarrow n^{\underline{th}} \text{ partial sum of the series}$$

$$\rightarrow \boxed{\{ S_1, S_2, S_3, \dots, S_n, \dots \}} \rightarrow \text{Sequence of partial sums}$$

If $\lim_{n\to\infty} S_n$ exists, then we say that the series converges and the sum of the series is:

$$\boxed{\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} S_n}$$

Otherwise, we say that the series diverges.

E.g. We will this formula to find:

$$\sum_{i=1}^{\infty} \frac{1}{2^i}.$$

Sol:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \lim_{n\to\infty} \boxed{S_n}$$

Step 1: Find the n^{th} partial sum S_n.

$$S_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{1}{2^i}$$

$$S_n = \frac{1}{2} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{8}} + \cdots + \cancel{\frac{1}{2^n}}$$

Subtract side by side

$$\frac{1}{2}\cdot S_n = \cancel{\frac{1}{4}} + \cancel{\frac{1}{8}} + \cdots + \cancel{\frac{1}{2^n}} + \frac{1}{2^{n+1}}$$

$$S_n - \frac{1}{2}S_n = \frac{1}{2} - \frac{1}{2^{n+1}}$$

$$\frac{1}{2}S_n = \frac{1}{2} - \frac{1}{2^{n+1}} \to \boxed{S_n = 1 - \frac{1}{2^n}}$$

We just found a formula for S_n, the n^{th} partial sum.

<u>Step 2</u>: Find $\lim\limits_{n\to\infty} S_n$.

$$\lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

Conclusion: $\boxed{\sum_{i=1}^{\infty} \frac{1}{2^i} = 1}$

Note: The general process for finding the sum of an infinite series consists of 2 steps.

Step 1: Find a formula for S_n, the n^{th} partial sum.

Step 2: Find $\lim_{n \to \infty} S_n$.

The series that we just considered is special, it is an example of a geometric series. For geometric series, there is an easy formula to find the sum if it exists, we don't have to go through the general process.

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

$$\times\frac{1}{2} \quad \times\frac{1}{2} \quad \times\frac{1}{2} \quad \times\frac{1}{2}$$

→ common ratio.

To get from one term to the next, we always multiply it by the same number. Any series with this property is called $\boxed{\text{geometric}}$. The constant that takes us from one term to the next is called the common ratio.

E.g. $s = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

$$\times\left(-\frac{2}{3}\right) \quad \times\left(-\frac{2}{3}\right) \quad \times\left(-\frac{2}{3}\right)$$

→ common ratio.

→ this series is also geometric.