Ex. Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges.

The function associated with this series is $f(x) = \frac{1}{\sqrt{x}}$.

(1) Positive on $[1, \infty)$ ✓ (3) Decreasing on $[1, \infty)$ ✓

(2) Continuous on $[1, \infty)$ ✓

$\longrightarrow$ Integral Test applies.

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{\sqrt{x}}\,dx = \lim_{t \to \infty} \int_1^{t} x^{-1/2}\,dx$$

$$= \lim_{t \to \infty} \left(2\sqrt{x}\right)\Big|_1^t$$

$$= \lim_{t \to \infty} \left(2\sqrt{t} - 2\right) = \infty$$

So, the integral diverges. Hence, the integral diverges.

So far, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges,

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Q: For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges?

Result: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{diverges if } p \leq 1 \\ \text{converges if } p > 1 \end{cases}$

Why? if $p \leq 0$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges. So we only need to consider the case $p > 0$.

The function associated with this series is $f(x) = \frac{1}{x^p}$.

$$f \begin{cases} \text{positive} \\ \text{continuous} \quad \text{on } [1,\infty) \\ \text{decreasing} \end{cases}$$

$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx$. By the p-integral result in 3.7.

$$\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{diverges if } p \leq 1 \\ \text{converges if } p > 1. \end{cases}$$

So does the series.

Estimate the Sum of a Series with the integral test

Suppose that we have used the integral test to determine that the series $\sum_{n=1}^{\infty} a_n$ converges.

We then use a partial sum $S_N = \sum_{n=1}^{N} a_n$ to approximate the sum of the infinite series, i.e., we add up the first N terms.

We call the infinite sum s. We want to know how far away S_N is from the infinite sum s.

Let $\boxed{R_N} = \boxed{s} - \boxed{S_N}$

N^{th} Remainder

sum of infinite series

N^{th} partial sum = finite sum of first N terms

→ want the bounds for the remainder R_N

The integral test says this:

$$\boxed{\underbrace{\int_{N+1}^{\infty} f(x)\,dx}_{\text{Lower bound for the remainder}} < R_N < \underbrace{\int_{N}^{\infty} f(x)\,dx}_{\text{The upper bound for the remainder}}}$$

Lower bound for the remainder

The upper bound for the remainder

E.g. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$

This series converges because it is a p-series with $p = 3 > 1$.

Suppose that we use $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$ to approximate the infinite sum.

$$S_{10} = 1 + \frac{1}{8} + \dots + \frac{1}{1000} \approx 1.1975$$

Q 1: Find the bounds for the error:

$$R_{10} = s - S_{10}$$

R_{10}: 10^{th} remainder

s: actual inf. sum

S_{10}: 10^{th} partial sum

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx < R_{10} < \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

We have $\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$

So, $\int_{10}^{\infty} \frac{1}{x^3}dx = \left(-\frac{1}{2x^2}\right)\Big|_{10}^{\infty} = 0 - \left(-\frac{1}{2\cdot(10)^2}\right)$

$= \frac{1}{200} = 0.005$

$\int_{11}^{\infty} \frac{1}{x^3}dx = \left(-\frac{1}{2x^2}\right)\Big|_{11}^{\infty} = 0 - \left(-\frac{1}{2(11)^2}\right)$

$= \frac{1}{242} \approx 0.004$

$\longrightarrow \quad 0.004 < R_{10} < 0.005$

Q2: How many terms should we use so that the error is no greater than 0.001 ?

(i.e. Find N such that when we use S_N to estimate s, the remainder R_N satisfies $\boxed{R_N} \leq 0.001$)

We know: $R_N < \int_N^\infty f(x)dx$. Here $f(x) = \frac{1}{x^3}$.

All we need is to require that $\boxed{\int_N^\infty \frac{1}{x^3}dx} < 0.001$.

As a result, $R_N < 0.001$.

$$\int_N^\infty \frac{1}{x^3}dx = \left(-\frac{1}{2x^2}\right)\Bigg|_N^\infty = \frac{1}{2N^2}$$

Want: $\frac{1}{2N^2} < 0.001 \rightarrow \frac{1}{0.002} < N^2$

$N^2 > 500 \rightarrow N > \sqrt{500} \approx 22.31$

So, we need to use 23 terms to ensure that $R_N < 0.001$.