5.3. The Divergent Test and the Integral Test

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1 The Divergent Test

Suppose we have a series $\sum_{n=1}^{\infty} \boxed{a_n}$.

If $\lim_{n \to \infty} a_n \neq 0$ or $\lim_{n \to \infty} a_n$ does not exist, then the series diverges.

E.g. $\sum_{n=1}^{\infty} \boxed{\frac{n^2}{5n^2+4}}$; $a_n = \frac{n^2}{5n^2+4}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$$

The Divergence Test says that this series diverges.

E.g. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\boxed{e^{1/n^2}}}_{a_n}$; $a_n = e^{1/n^2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\boxed{1/n^2} \to 0} = e^{0} = 1 \neq 0$$

→ The series diverges by the divergence test.

E.g. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\boxed{\cos\left(\frac{1}{n^4}\right)}}_{a_n}$; $a_n = \cos\left(\frac{1}{n^4}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\Big(\underbrace{\boxed{\tfrac{1}{n^4}}}_{\to 0}\Big) = \cos(0) = 1 \neq 0$$

→ The series diverges by the divergence test.

Warning The divergence test cannot be used to test for convergence; i.e., if $\lim_{n\to\infty} a_n = 0$, it does NOT imply that the series converges.

For e.g., $\sum_{n=1}^{\infty} \boxed{\frac{1}{n}}$; $a_n = \frac{1}{n}$; $\lim_{n\to\infty} a_n = 0$

$\sum_{n=1}^{\infty} \boxed{\frac{1}{n^2}}$; $a_n = \frac{1}{n^2}$; $\lim_{n\to\infty} a_n = 0$

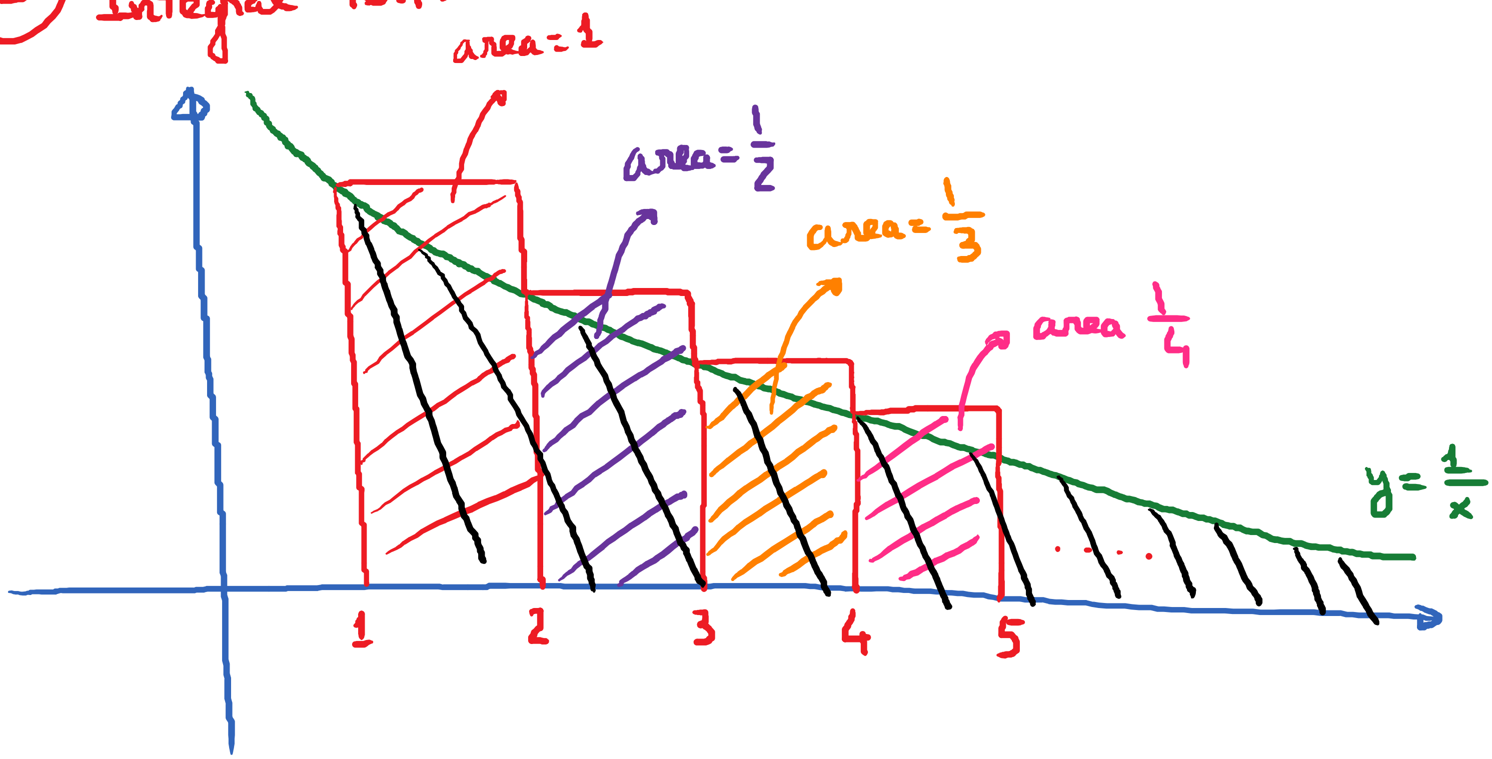
Later, we will see that $\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$ diverges and

harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

→ The Divergence Test fails here, you cannot use it when the limit is 0.

② Integral Test.



$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

= Sum of areas of the shaded rectangles

$$> \underbrace{\text{Area under the curve } y = \frac{1}{x};\ x \geq 1}_{}$$

$$= \int_1^{\infty} \frac{1}{x}\,dx$$

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$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \ln(x) \Big|_1^t$$

$$= \lim_{t\to\infty} \ln(t) = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

The Integral Test:

Suppose $s = \sum_{n=1}^{\infty} a_n$ is a series. $a_n = f(n)$

(a_n is given by a function of n)

We can apply the integral test if all the following conditions are satisfied:

(1) f is positive on $[N,\infty)$ for some $N \geq 1$.

(2) f is continuous on $[N,\infty)$ for some $N \geq 1$.

(3) f is decreasing on $[N,\infty)$ for some $N \geq 1$.

$\left(f'(x) < 0 \text{ on } [N,\infty) \right)$

The integral test then says:

If $\int_N^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.

If $\int_N^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

E.g. Consider the series: $\sum_{n=1}^{\infty} \boxed{\frac{1}{n^2}}$.

Use The Integral Test to determine whether this series converges or diverges.

$a_n = \frac{1}{n^2} \longrightarrow f(n) = \frac{1}{n^2}$

So, $f(x) = \frac{1}{x^2}$ is the function associated with this series.

① Is f positive on $[1, \infty)$? Yes.

② Is f continuous on $[1, \infty)$? Yes.

③ Is f decreasing on $[1, \infty)$? Yes.

$\left(f'(x) = -\frac{2}{x^3} < 0 \text{ on } [1, \infty) \right)$

So, the integral test applies:

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^t x^{-2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Bigg|_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) = 1$$

So, the integral converges.

So, the series converges.

Note: The value of the integral has nothing to do with the sum of the series. It can only tell you that the series converges but it does not tell you what the series converges to.