Tuesday, October 23, 2018 9:13 AM

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ will behave in the exact same way; i.e., they both converge or they both diverge.

E.g. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$.

Scratch work: when n is large, the series behave like:

$$\sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{2n^2}{n^{5/2}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

diverges b/c p series; $p = \frac{1}{2}$.

Real work: Use the limit comparison test

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$$ Limit comparing it to $$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

a_n $\qquad$ b_n

So, $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{n^5+5}} \cdot n^{1/2}$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{\sqrt{n^5+5}} = \boxed{2} \rightarrow L$$

$L > 0$. So, the limit comparison test says that

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$$ and $$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}}$$ behave in the same way.

Since $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges $\left(p=\frac{1}{2}\right)$; the original series diverges.

Ex. Determine whether the given series converges or diverges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+2n^2-1}{6n^6+4n^4+1}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{4}{n}\right)}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n-n}$

(a) Limit comparing this to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ b_n

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4+2n^2-1}{6n^6+4n^4+1} \cdot n^2 = \lim_{n\to\infty} \frac{n^6+2n^4-1}{6n^6+4n^4+1}$$

$= \frac{1}{6} > 0$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the original series converges

ⓒ Compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ b_n

$0 \cdot \infty$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{4}{n}\right)}{n} \cdot n^2 = \lim_{n\to\infty} \left[\ln\left(1+\frac{4}{n}\right)\right] \cdot n$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{4}{n}\right)}{\frac{1}{n}} \left(\frac{0}{0}\right) =$$

L'Hopital Rule

$$\lim_{n\to\infty} \frac{\boxed{\dfrac{\cancel{-}\frac{4}{n^2}}{1+\frac{4}{n}}}}{\boxed{\cancel{-}\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{\frac{4}{n^2}}{1+\frac{4}{n}} \cdot n^2 = \lim_{n\to\infty} \frac{4}{1+\frac{4}{n}}$$

$$= 4 > 0$$

So, the original series converges.

(d) Comparing to $\sum_{n=1}^{\infty} \frac{1}{3^n}$ b_n

$$\lim_{n\to\infty} \frac{1}{3^n - n} \cdot 3^n = \lim_{n\to\infty} \frac{3^n}{3^n - n} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{3^n \cdot \ln(n)}{3^n \cdot \ln(n) - 1} = 1 > 0$$

So, the series converges because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is geometric

with common ratio $\frac{1}{3} < 1$.
