5.4. Comparison Tests and Limit Comparison Tests

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Direct Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $a_n \leq b_n$ for every $n \geq N$, then:

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

In short, if your series is smaller than a convergent series, then it converges. If it is greater than a divergent series, then it diverges.

E.g. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Compare this series with an appropriate p-series to determine whether it converges or diverges.

Idea: Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

We know: $n^2+1 > n^2$ for every $n \geq 1$

So, $\displaystyle\frac{1}{n^2+1} < \frac{1}{n^2}$ for every $n \geq 1$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2}}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p>1$),

the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ must converge by the comparison test.

E.g. Determine whether the given series converges or diverges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n^3 + 3n^2 + n + 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

Sol: (a) $5n^3 + 3n^2 + n + 1 > 5n^3$; for every $n \geq 1$

So, $\displaystyle\frac{1}{5n^3 + 3n^2 + n + 1} < \frac{1}{5n^3}$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n^3 + 3n^2 + n + 1} \leq \sum_{n=1}^{\infty} \frac{1}{5n^3} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n^3}$

converges; p-series with $p = 3$.

Hence, this series converges.

(b) $5^n - 1 < 5^n$ for every $n \geq 1$

So, $\dfrac{6^n}{5^n - 1} > \dfrac{6^n}{5^n} = \left(\dfrac{6}{5}\right)^n$

So, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}} \geq \displaystyle\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$

diverges b/c it is greater than a divergent series.

geometric series with common ratio $= \dfrac{6}{5} > 1$

So, diverges.

(c) $\dfrac{\ln(n)}{n} > \dfrac{1}{n}$ for $n \geq 3$

(Reason: $\ln(n) > 1$ for $n \geq 3$)

So, $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n)}{n}} \geq \boxed{\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}}$ → diverges, p-series $p = 1$.

→ diverges

d $\ln(n) < n$ for $n \geq 2$

So, $\dfrac{1}{\ln(n)} > \dfrac{1}{n}$ for $n \geq 2$

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n}$$ diverges

diverges

Limit Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms

The limit comparison test says that:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where L is a finite and positive number, i.e., $0 < L < \infty$, then