5.5. Alternating Series Test

A.S.T.

Given an alternating series of the form $\sum_{n=1}^{\infty} (-1)^n \cdot b_n$

or $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n \geq 0$.

If the sequence (b_n) satisfies the conditions:

① $b_n \geq b_{n+1}$ for all $n \geq N$

(it is a decreasing sequence)

② $\lim_{n\to\infty} b_n = 0$

Then the alternating series converges.

E.g. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\underbrace{\boxed{\frac{1}{n}}}_{b_n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

① Is b_n decreasing? Yes (b/c $b_n = \frac{1}{n}$)

② Is $\lim_{n\to\infty} b_n = 0$? Yes ($\lim_{n\to\infty} \frac{1}{n} = 0$)

The A.S.T says that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges

E.g. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\cdot\frac{1}{n!}$ → factorial.

$n! = n(n-1)(n-2)\cdots 1$

$2! = 2\cdot 1 = 2$

$3! = 3\cdot 2\cdot 1 = 6$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$0! = 1$

$b_n = \frac{1}{n!}$

① Decreasing? Yes.

② $\lim_{n\to\infty}\frac{1}{n!} = 0$? Yes.

By the A.S.T. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n!}$ converges.

E.g. $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \underbrace{\boxed{\frac{n^2}{2n^3+1}}}_{b_n}$ → Alternating Series

$$b_n = \frac{n^2}{2n^3+1}$$

1. Is b_n decreasing?

$b_n = \frac{n^2}{2n^3+1}$ → function associated with b_n is

$$f(n) = \frac{n^2}{2n^3+1}$$

quotient rule

$$f'(n) = \frac{(2n^3+1)2n - n^2(6n^2)}{(2n^3+1)^2} = \frac{4n^4+2n-6n^4}{(2n^3+1)^2}$$

$$= \frac{2n-2n^4}{(2n^3+1)^2} = \boxed{\frac{2n(\underbrace{\boxed{1-n^3}}_{\leq 0})}{(2n^3+1)^2}} \leq 0 \text{ when } n \geq 1$$

So, $f(n)$ is decreasing on $[1, \infty)$.

Hence, b_n is a decreasing sequence.

(2) Is $\lim_{n \to \infty} b_n = 0$?

$$\lim_{n \to \infty} \frac{n^2}{2n^3 + 1} = 0 \quad (\text{deg top} < \text{deg bottom})$$

By A.S.T., the series converges.

The concepts of Absolute convergence and Conditional Convergence.

E.g. Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$

By the A.S.T. this series converges.

Consider the "absolute value series" of the above series:

It will be the series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

By the p-series test, this series diverges. $(p = 1)$

→ We say that the original series $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ converges condinally.

Conditional Convergence means $\underbrace{\sum_{n=1}^{\infty} a_n}$ converges but → original converges

$\underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{absolute value does not converges}}$ does not converge