$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1.$$ Diverges.

Why is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$?

$$L = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$\ln(L) = \lim_{n\to\infty} n\cdot\ln\left(1+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$ $\left(\frac{0}{0}\right) \to$ L'Hopital

$$= \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{-\frac{1}{n^2}}} = \lim_{n\to\infty}\frac{1}{1+\underbrace{\frac{1}{n}}_{0}} = 1$$

$L = e^1 = e.$

Note: Have lots of factorials in series → consider ratio test.

Root Test Given a series $\sum_{n=1}^{\infty} a_n$.

Calculate the limit: $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ $\left(|a_n|^{\frac{1}{n}}\right)$

Let $p = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

Case 1: $0 \le p < 1$: then the series converges absolutely.

Case 2: $p > 1$: the series diverges

Case 3: $p = 1$: Test fails.

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E.g. Consider the series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^{n}$.

Converges or diverges?

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^{n}} = \lim_{n\to\infty}\left(\frac{2n+3}{3n+2}\right) = \frac{2}{3} < 1$$

$\rightarrow$ Series converges absolutely.

E.g. $\sum_{n=3}^{\infty}\frac{n^n}{[\ln(n)]^n}$.

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{[\ln(n)]^n}} = \lim_{n\to\infty}\frac{n}{\ln(n)} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty}\frac{1}{\frac{1}{n}} = \lim_{n\to\infty}(n) = \infty > 1. \text{ Diverges}$$

Ex. Determine all the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

converges

Ratio Test: $a_n = \frac{x^n}{n!}$; $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \frac{|x|}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

Series converges for all x.