5.6. Ratio Test and Root Test

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Ratio Test

Given a series $\sum_{n=1}^{\infty} a_n$.

Calculate the limit $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

Let $p = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

The Ratio Test says this:

Case 1: $0 \leq p < 1$, then the series converges.

Moreover, it converges absolutely; i.e., the "absolute value series" converges.

Case 2: $p > 1$, then the series diverges.

Case 3: $p = 1$, the test fails; i.e., it does not provide any info. about the series.

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a_n

E.g. Consider the series $\displaystyle\sum_{n=1}^{\infty} \boxed{(-1)^n \cdot \frac{n^3}{3^n}}$.

Convergence or Divergence?

Apply the Ratio Test. Find $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$a_n = (-1)^n \cdot \frac{n^3}{3^n} \quad ; \quad a_{n+1} = (-1)^{n+1} \cdot \frac{(n+1)^3}{3^{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{\cancel{n+1}} \cdot \frac{(n+1)^3}{3^{\cancel{n}+1}} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^n} \cdot n^3}\right|$$

$$= \left|\frac{(-1)\cdot(n+1)^3}{3 \cdot n^3}\right| = \frac{(n+1)^3}{3n^3}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \boxed{\frac{1}{3}} < 1$$

The series converges absolutely.

Ex. Use the Ratio Test to determine convergence or divergence

ⓐ $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(n!)^2}{(2n)!}$ (Recall: $n! = n(n-1)(n-2)\ldots 2$)

ⓑ $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Sol: ⓐ $a_n = (-1)^n \cdot \dfrac{(n!)^2}{(2n)!}$; $a_{n+1} = (-1)^{n+1} \cdot \dfrac{[(n+1)!]^2}{[2(n+1)]!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(-1)^{\cancel{n+1}} \cdot [(n+1)!]^2}{\boxed{(2n+2)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{(-1)^n} \cdot (n!)^2} \right|$$

$(2n+2)(2n+1)[\cancel{(2n)!}]$

$$= \left| \frac{(-1)\left[(n+1)\cdot(n!)\right]^2}{(2n+2)(2n+1)} \cdot \frac{1}{(n!)^2} \right|$$

$$= \frac{(n+1)^2 \cdot \cancel{(n!)^2}}{(2n+2)(2n+1)\cdot\cancel{(n!)^2}} = \frac{(n+1)^2}{(2n+2)(2n+1)}$$

Then: $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$

Series converges absolutely.

(b) $a_n = \frac{n^n}{n!}$; $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{\cancel{n!}}{n^n} = \frac{(n+1)^{n\cancel{+1}}}{\cancel{(n+1)}\cdot n^n}$$

$(n+1)\cdot(\cancel{n!})$