6.1. Power Series

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What is a power series?

A power series centered at 0 is a series of the form

$$\boxed{\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots}$$

(Compare: polynomial $\underbrace{c_0 + c_1 x + c_2 x^2 + \dots + c_N x^N}_{\text{polynomial}}$)

The $c_1, c_2, c_3, \dots$ are constants and they are called the coefficients of the series.

E.g. If we take $c_0 = c_1 = c_2 = c_3 = \dots = 1$; i.e., $c_n = 1$ for every $n \geq 0$, then the series $\sum_{n=0}^{\infty} c_n x^n$ looks like:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$\cdot x \quad \cdot x \quad \cdot x$

$\longrightarrow$ a geometric series with common ratio x.

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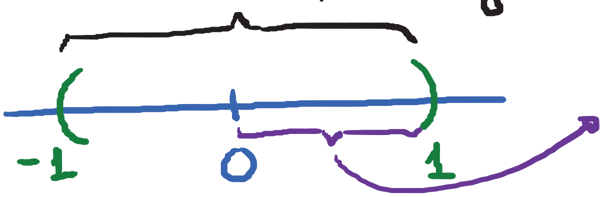
If $|x| < 1$; i.e., $-1 < x < 1$, the series above converges.

Moreover,
$$\boxed{\underbrace{\sum_{n=0}^{\infty} x^n}_{} = \frac{1}{\underbrace{1-x}_{}} \ ; \quad -1 < x < 1}$$

series = function

Interval of convergence (I.O.C)



Radius of convergence (R.O.C) = 1

center of series

E.g. Consider the power series

$$\sum_{n=0}^{\infty} \boxed{n!}\, x^n$$

(Here $c_n = n!$
So, $c_0 = 1$; $c_1 = 1$; $c_2 = 2$;
$c_3 = 6$; $c_4 = 24$, ...)

$(1 + x + 2x^2 + 6x^3 + 24x^4 + \dots)$

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Q: For what values of x does this series converge?

→ Apply the Ratio Test

① $\left| \dfrac{a_{n+1}}{a_n} \right| = \left| \dfrac{(n+1)!\, x^{\cancel{n}+1}}{\cancel{n!}\, \cancel{x^n}} \right| = (n+1) \cdot |x|$

② $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} (n+1)|x| = \infty > 1$

except when $x = 0$; limit $= 0 < 1$

Conclusion: The series diverges for all values of $x \neq 0$.

The only value of x for which this series converges is $x = 0$.

E.g. Consider the power series $\displaystyle\sum_{n=0}^{\infty} \underbrace{\boxed{\frac{(-1)^n \cdot n}{4^n}}}_{c_n} x^n$

$$\left(c_n = \frac{(-1)^n \cdot n}{4^n} \right)$$

Q: Determine the values of x for which this series converges.

Ratio test: (1) $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{(-1)^{n+1}} \cdot (n+1) \cdot x^{\cancel{n+1}}}{4^{\cancel{n+1}}} \cdot \frac{\cancel{4^n}}{\cancel{(-1)^n} \cdot n \cdot \cancel{x^n}}\right|$

$$= \frac{n+1}{4n} \cdot |x|$$

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② $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\underbrace{\boxed{\frac{n+1}{4n}}}_{\to \frac{1}{4}}\cdot|x| = \frac{|x|}{4}$

If $\frac{|x|}{4} < 1$, the Ratio test implies that series converges.

$\frac{|x|}{4} < 1 \leftrightarrow |x| < 4 \leftrightarrow -4 < x < 4.$

So, the series converges for all values of x in $(-4, 4)$

Recall: Ratio test fails when limit of ratio $= 1$; i.e.,

$\frac{|x|}{4} = 1$; i.e., $x = 4$ or $x = -4$.

→ We do need to test the series when $x = 4$ or $x = -4$ separately to see whether it converges or not.

* Plug $\boxed{x = 4}$ into series: Series becomes $\sum_{n=0}^{\infty}\frac{(-1)^n\cdot n}{4^n}\left(\underset{x}{4}\right)^n$

$= \sum_{n=0}^{\infty}\frac{(-1)^n\cdot n\cdot \cancel{4^n}}{\cancel{4^n}} = \sum_{n=0}^{\infty}(-1)^n\cdot n$ diverges by the Divergence Test.

* Plug $x=-4$ into the series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{4^n}(-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{\cancel{4^n}}(-1)^n \cdot \cancel{4^n} = \sum_{n=0}^{\infty} n$$

diverges obviously.

Conclusion: Interval of convergence: $(-4, 4)$

R.O.C. $= 4$

E.g. Determine the values of x for which the series converges. Find I.O.C and R.O.C.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ (Centered at 3)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}$ (Centered at 6)

Note:

$$\sum_{n=0}^{\infty} c_n x^n \longrightarrow \text{centered at } 0$$

$$\sum_{n=0}^{\infty} c_n (x-a)^n \longrightarrow \text{centered at } a.$$

Sol (a) Ratio test ① $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{(x-3)^n}}\right|$

$$= \frac{n}{n+1} \cdot |x-3|$$

② $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \underbrace{\boxed{\frac{n}{n+1}}}_{\to 1} |x-3| = |x-3|$

If $|x-3| < 1$, the series converges.

$$|x-3| < 1 \longleftrightarrow -1 < x-3 < 1$$

$$\longleftrightarrow 2 < x < 4$$