③ $f(x) = \cos x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{on } (-\infty, \infty)$$

Important Series.

Function	Series	I.O.C
$f(x) = e^x$	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$f(x) = \sin x$	$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	$(-\infty, \infty)$
$f(x) = \cos x$	$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	$(-\infty, \infty)$
$f(x) = \frac{1}{1-x}$	$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$f(x) = \ln(1+x)$	$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$	$(-1, 1)$

$f(x) = \ln(1+x)$	$\ln(1+x) = \sum_{n=1} \cdots \frac{\cdots}{n} = x - \frac{\cdots}{2} \cdots \frac{\cdots}{3}$	
$f(x) = \arctan(x)$	$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots$	$(-1, 1)$

Taylor and Maclaurin Polynomials.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$P_0(x) = 1 \leftarrow 0^{\underline{th}}$ degree Maclaurin polynomial for e^x

$P_1(x) = 1 + x \leftarrow 1^{\underline{st}}$ degree ______

$P_2(x) = 1 + x + \frac{x^2}{2!} \leftarrow 2^{\underline{nd}}$ degree ______

$\vdots$

$P_5(x) = \boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}} \longleftrightarrow 5^{\underline{th}}$ degree poly.

Approximate $e^{1.5}$ using $P_5(x)$:

$P_5(\underline{1.5})$

4.46172

Definition of the n^{th} degree Taylor polynomial for a function f.

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$\boxed{T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j}$$

n^{th} degree Maclaurin polynomial

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!}x^j$$

Taylor Remainder Theorem

Assume f is differentiable $(n+1)$ times on an interval I containing the point a.

Let $T_n(x) = n^{\text{th}}$ degree Taylor polynomial for f centered at a.

Let $$\boxed{\underbrace{R_n(x)}_{n^{\text{th}}\text{ Taylor Remainder}} = \underbrace{f(x)}_{\text{function}} - \underbrace{T_n(x)}_{\text{approximation}}}$$

Taylor Remainder Theorem:

$$|R_n(x)| \le \underbrace{\boxed{\frac{\boxed{M}}{(n+1)!}|x-a|^{n+1}}}_{\text{upper bound for error}}$$

M is an upper bound for $|f^{(n+1)}(x)|$ on I; i.e., M is such that $|f^{(n+1)}(x)| \leq M$ for all x in I.

E.g. $f(x) = \sqrt{x}$

1. Find the 1^{st} and 2^{nd} degree Taylor polynomial for f centered at $a = 4$.
2. Use $T_1(x)$ and $T_2(x)$ to estimate $\sqrt{6}$.
3. Find upper bounds for $R_1(6)$ and $R_2(6)$

$$T_1(x) = \boxed{c_0} + \boxed{c_1}(x-4)$$

$$c_0 = f(4) = \sqrt{4} = 2$$

$$c_1 = f'(4) = \frac{1}{4}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

$$\boxed{T_1(x) = 2 + \frac{1}{4}(x-4)}$$

$$T_2(x) = c_0 + c_1(x-4) + c_2(x-4)^2$$

$$c_2 = \frac{f''(4)}{2!}$$

$$f''(x) = -\frac{1}{4\sqrt{x^3}}$$

So, $c_2 = -\frac{1}{64}$

$$\boxed{T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2}$$

1st approximation

(b) $T_1(6) = 2 + \frac{1}{4}(6-4) = \boxed{2.5}$

2nd approx.

$T_2(6) = 2 + \frac{1}{4}(6-4) - \frac{1}{64}(6-4)^2 = \boxed{2.4375}$

(c) Upper bounds for $R_1(6)$ and $R_2(6)$

Taylor's Remainder Theorem:

$$|R_1(6)| \leq \frac{M}{2!}|6-4|^2 = 2M$$

M is an upper bound for $|f''(x)|$ on $[4,6]$

$$|f''(x)| = \frac{1}{4|x|^{3/2}}.$$ We can take M to be

$$|f''(4)| = \frac{1}{4|4|^{3/2}} = \frac{1}{32}$$

So, $|R_1(6)| \leq 2 \cdot \frac{1}{32} = \frac{1}{16}$

$$|R_2(6)| \le \frac{M}{3!}|6-4|^3$$

M: upper bound for $|f'''(x)|$ on $[4,6]$

$$|f'''(x)| = \frac{3}{8|x|^{5/2}}.$$ We can take M to be:

$$|f'''(4)| = \frac{3}{8(4)^{5/2}} = \frac{3}{256}$$

So, $$|R_2(6)| \le \frac{3}{256}\cdot\frac{1}{6}\cdot(2)^3 = \frac{1}{64}$$