6.3. Taylor Series and Maclaurin Series

Tuesday, November 13, 2018 8:08 AM

Recall: $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \ ; \ |x| < 1$

Integrate $\rightarrow$ $\boxed{\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n} \ ; \ |x| < 1}$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \ ; \ |x| < 1$

Integrate $\rightarrow$ $\boxed{\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \ ; \ |x| < 1}$

Point: Start with a geometric series $\rightarrow$ integrate / differentiate

$\rightarrow$ series for many functions

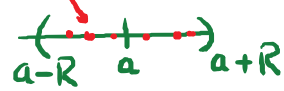
What is the series for e^x ?

$\rightarrow$ Taylor and Maclaurin Series.

Taylor Theorem:

If a function f has a power series expansion centered at a; that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \; ; \qquad |x-a| < R$$



Then the coefficients c_n of the series are given by the formula:

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

$f^{(n)}(a)$: n^{th} derivative of f evaluated at $x = a$.

In other words, the Taylor series for $f(x)$ is given by

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}}$$

In the special case that the center $a=0$, the Taylor series for $f(x)$ is called the Maclaurin series for $f(x)$.

So, the Maclaurin series for $f(x)$ is:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Note: Maclaurin = Taylor with center = 0

E.g. (a) Find the Maclaurin series for the given function.

(Hint: n^{th} coeff. $c_n = \dfrac{f^{(n)}(0)}{n!}$; take the first few derivatives of function and find the pattern for $f^{(n)}(0)$)

(b) Find I.O.C and R.O.C of the series.

Function: (1) $f(x) = e^x$

(2) $f(x) = \sin x$

(3) $f(x) = \cos x$

① $f(x) = e^x$

$$e^x = \sum_{n=0}^{\infty} c_n x^n .$$

Formula for c_n is $c_n = \dfrac{f^{(n)}(0)}{n!}$

* $\boxed{c_0 = f(0) = 1}$

* $c_1 = \dfrac{f'(0)}{1!} = f'(0)$

So, $\boxed{c_1 = 1}$

* $c_2 = \dfrac{f''(0)}{2!}$

So, $\boxed{c_2 = \dfrac{1}{2!}}$

* $c_3 = \dfrac{f'''(0)}{3!}$

So, $\boxed{c_3 = \dfrac{1}{3!}}$

$f'(x) = e^x \to f'(0) = 1$

$f''(x) = e^x \to f''(0) = 1$

$f'''(x) = e^x \to f'''(0) = 1$

So, in general, $\boxed{c_n = \frac{1}{n!}}$

So, $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

(b) I.O.C and R.O.C ?

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{x^{n+1}}}{(n+1)!} \cdot \frac{n!}{\cancel{x^n}}\right|$

$$= \frac{|x|}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

So, series converges for all values x.

Important Result:

$$\boxed{\underbrace{e^x}_{\text{function}} = \underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}_{\text{series}} \,; \quad \text{on } (-\infty, \infty)}$$

function = series for all value of x.

Application: $$e^{1.5} = \sum_{n=0}^{\infty} \frac{(1.5)^n}{n!}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \frac{f^{(n)}(0)}{n!}$
0	$\sin x$	$\sin(0) = 0$	$c_0 = 0$
1	$\cos x$	$\cos(0) = 1$	$c_1 = 1$
2	$-\sin x$	$-\sin(0) = 0$	$c_2 = 0$
3	$-\cos x$	$-\cos(0) = -1$	$c_3 = \frac{-1}{3!}$
4	$\sin x$	$\sin(0) = 0$	$c_4 = 0$
			$c_5 = \frac{1}{5!}$
			$c_6 = 0$
			$c_7 = \frac{-1}{7!}$

$$c_{2n} = 0 \;; \quad c_{2n+1} = \frac{(-1)^n}{(2n+1)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

(b) I.O.C and R.O.C

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}}\right|$

$$= \frac{|x|^2}{(2n+2)(2n+3)}$$

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

It converges for all x in $(-\infty, \infty)$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{on } (-\infty, \infty)}$$