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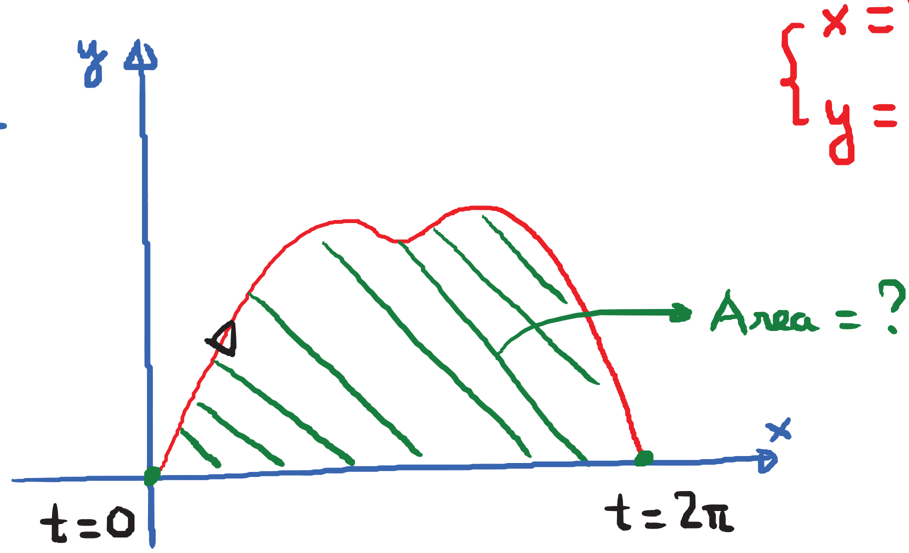
$$\text{Shaded Area} = \int_{x_1}^{x_2} y\,dx = \int_{t=a}^{t=b} y(t)\,|x'(t)|\,dt$$

$$\frac{dx}{dt} = x'(t) \longrightarrow dx = |x'(t)|\,dt$$

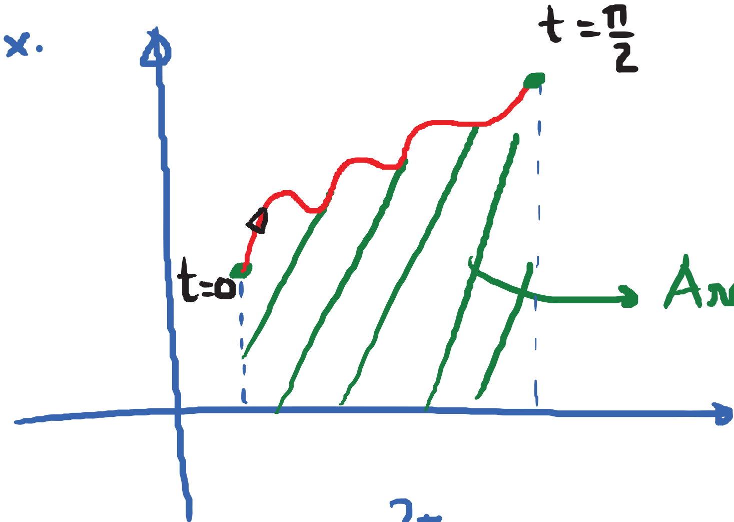
Ex.

$$\begin{cases} x = t - \sin(t) \\ y = 1 - \cos(t) \end{cases}$$

$$0 \le t \le 2\pi$$



Ex.



$$\begin{cases} x = \cos(t) \\ y = e^t \end{cases}$$

$$0 \le t \le \frac{\pi}{2}$$

Sol: Ex. $A = \int_0^{2\pi} y(t)\, x'(t)\, dt = \int_0^{2\pi} (1 - \cos(t)) \cdot (1 - \cos(t))\, dt$

$$A = \int_0^{2\pi} (1 - \cos(t))^2\, dt = \int_0^{2\pi} \Big(1 - 2\cos(t) + \underbrace{\cos^2(t)}_{\text{power reduction}}\Big)\, dt$$

$$= \int_0^{2\pi} \left(1 - 2\cos(t) + \boxed{\frac{1 + \cos(2t)}{2}}\right) dt$$

$$= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos(t) + \frac{1}{2}\cos(2t)\right) dt$$

$$= \left(\frac{3}{2}t - 2\sin(t) + \frac{1}{4}\sin(2t)\right)\Big|_0^{2\pi} = \boxed{3\pi}$$

Ex. 2. $A = \int_0^{\pi/2} \underbrace{e^t}_{y} \cdot \left|\underbrace{(-\sin(t))}_{x'} dt\right| = \ominus \int_0^{\pi/2} e^t \sin(t)\, dt$

$$\begin{cases} u = \sin(t) \\ dv = e^t\, dt \end{cases} \longrightarrow \begin{cases} du = \cos(t) \\ v = e^t \end{cases}$$

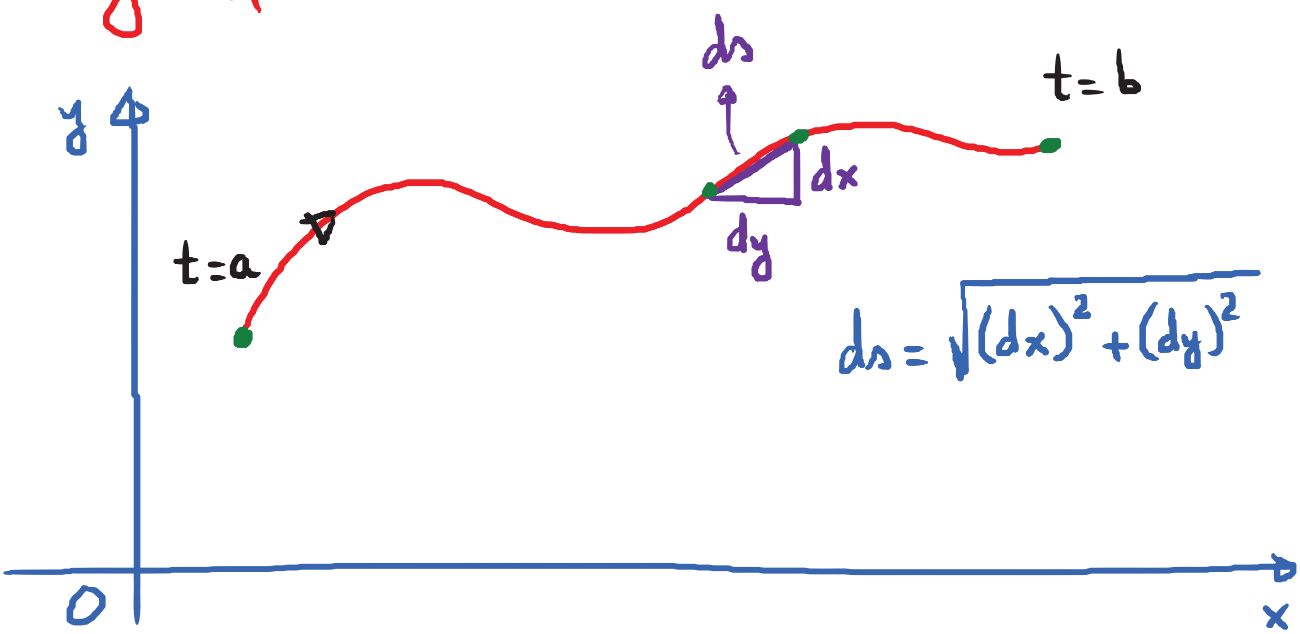
$$A = -e^t \sin t \Big|_0^{\pi/2} + \boxed{\int_0^{\pi/2} e^t \cos(t)\, dt}$$

$$\begin{cases} u = \cos t \\ dv = e^t dt \end{cases} \rightarrow \begin{cases} du = -\sin t \\ v = e^t \end{cases}$$

$$A = -e^{\pi/2} + e^t \cos(t)\Big|_0^{\pi/2} - \boxed{\int_0^{\pi/2} e^t(-\sin t)\, dt}^{\,A}$$

$$2A = -e^{\pi/2} - 1 \rightarrow A = \left| \frac{-e^{\pi/2} - 1}{2} \right|$$

Arc length of Parametric Curves.



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a \leq t \leq b.$$

L = length of curve from t = a to t = b

$$L = \int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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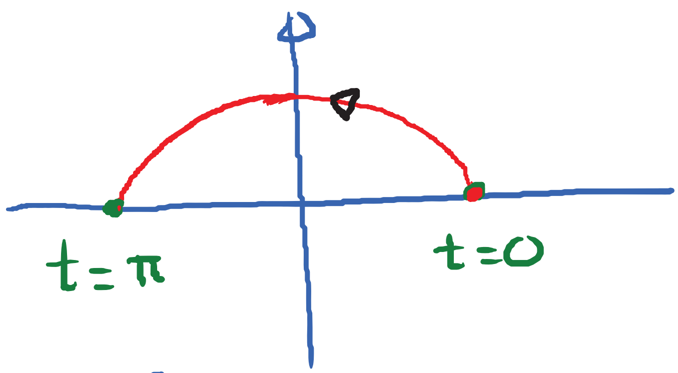
$$\boxed{\text{Arc length } L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

E.g. Find the length of the curve given by:

$$\begin{cases} x = 3\cos(t) \\ y = 3\sin(t) \end{cases} \quad 0 \le t \le \pi.$$



$$L = \int_0^\pi \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}\, dt$$

$$L = \int_0^\pi \sqrt{9\sin^2 t + 9\cos^2 t}\, dt = \int_0^\pi 3\, dt = \boxed{3\pi}$$

Ex. Find the length of the curve given by:

$$\begin{cases} x = 3t^2 \\ y = 2t^3 \end{cases} \quad 1 \le t \le 3$$

$$L = \int_1^3 \sqrt{(6t)^2 + (6t^2)^2}\, dt = 6\int_1^3 \sqrt{t^2 + t^4}\, dt$$

$$= 6\int_1^3 \sqrt{t^2(1+t^2)}\, dt = \frac{6}{2}\int_1^3 2t\sqrt{1+t^2}\, dt$$

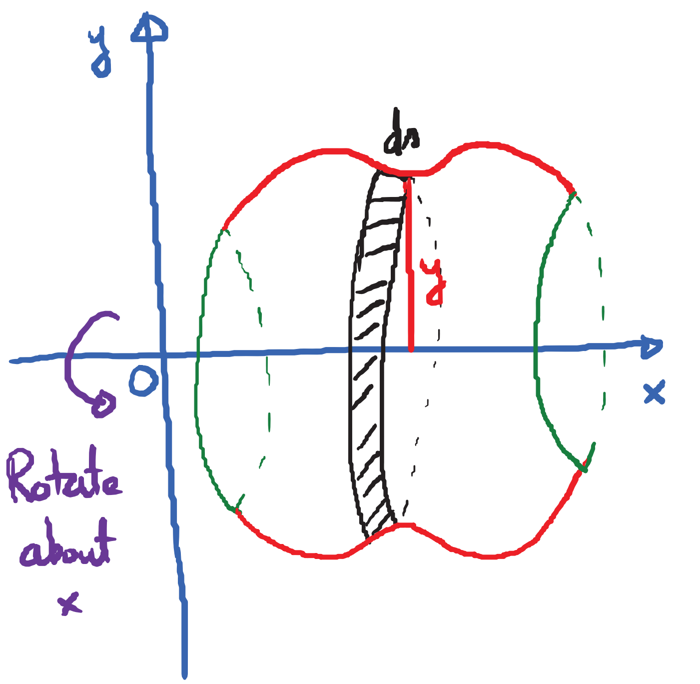
$u = 1 + t^2 \longrightarrow du = 2t\,dt$. $t = 1 \to u = 2$

$t = 3 \to u = 10$

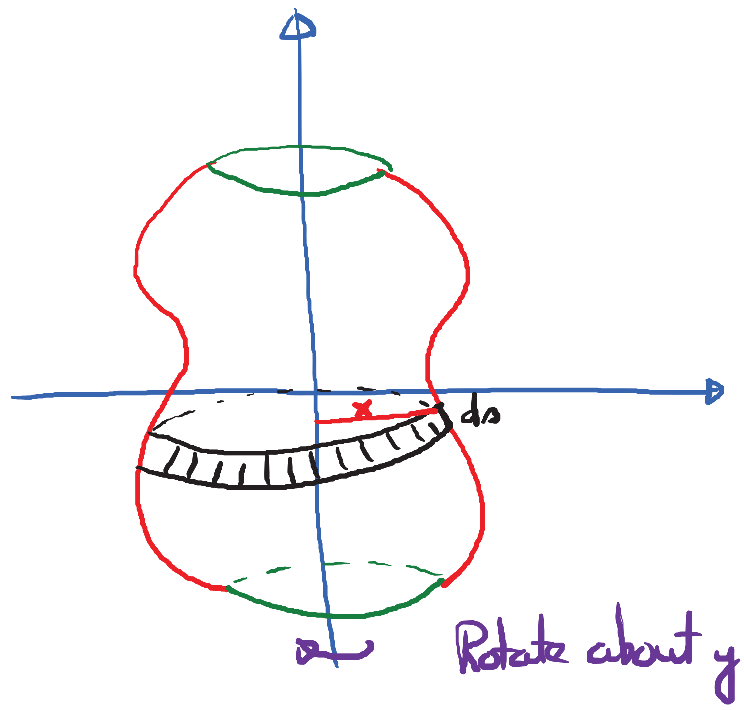
$$3\int_2^{10} \sqrt{u}\, du = \cancel{3}\cdot\frac{2u^{3/2}}{\cancel{3}}\Bigg|_2^{10} = 2\left[(10)^{3/2} - (2)^{3/2}\right]$$

$$= \boxed{2\left[10\sqrt{10} - 2\sqrt{2}\right]}$$

Surface Area



Surface area $= \int 2\pi y \, ds$



Surface area $= \int 2\pi x \, ds$

Curve is given as: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $0 \le t \le a$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Rotate about x :

$$\boxed{S = 2\pi \int_{t=a}^{t=b} y(t) \cdot \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

Rotate about y :

$$\boxed{S = 2\pi \int_{t=a}^{t=b} x(t) \cdot \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$