7.1 and 7.2 Parametric curves and Calculus of Parametric Curves

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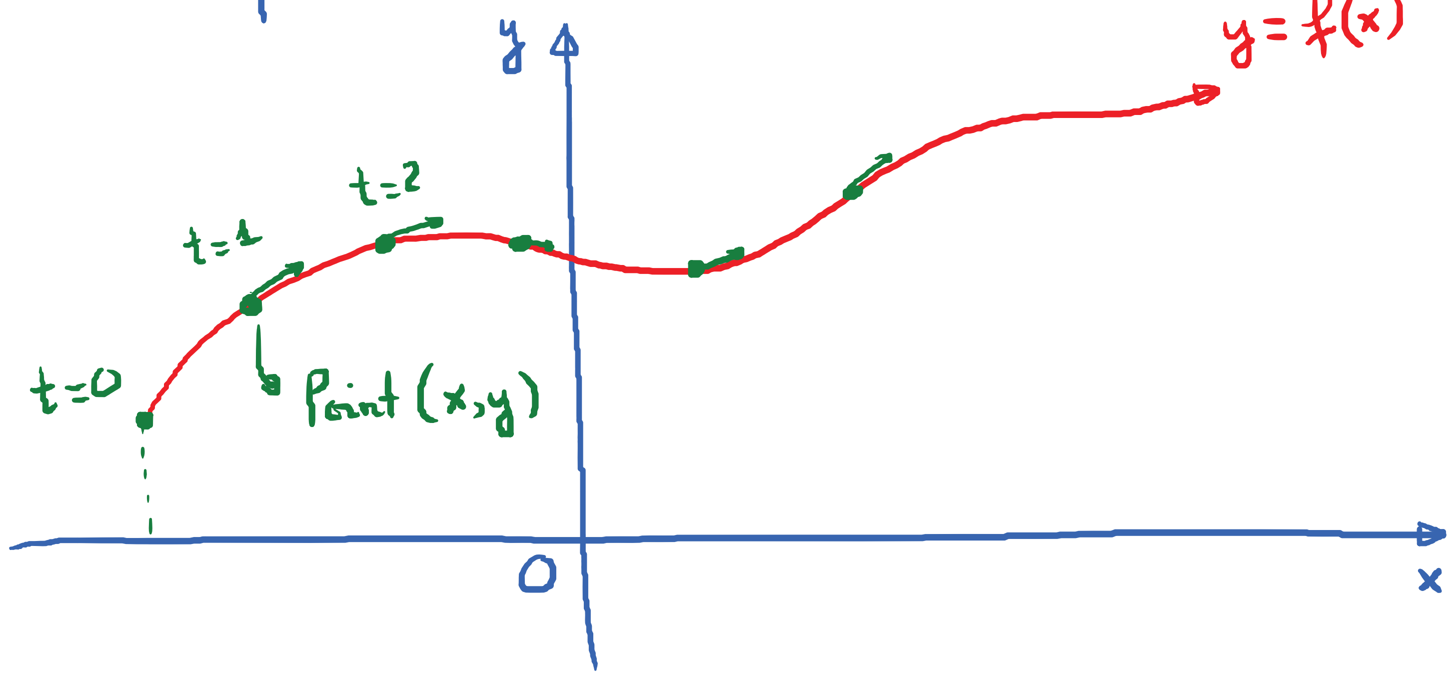
What is a parametric curve?



x and y coordinates of a point moving along on this curve change with respect to time.
Introduce the variable t for time.

→ Both x and y are functions of time t.

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

These 2 equations are called the parametric equations for the curve $y = f(x)$

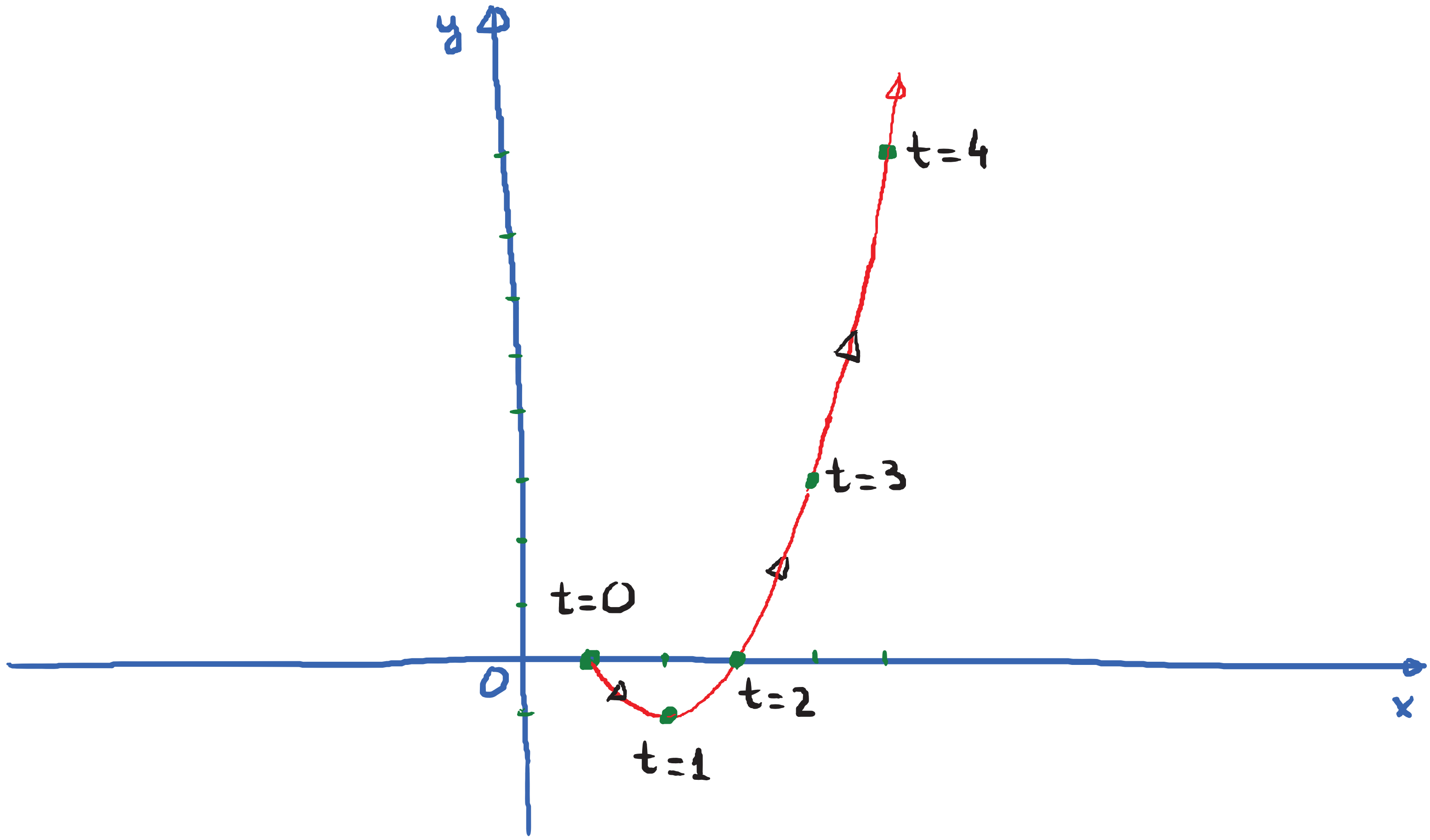
Eg. Given the parametric equations

$$\begin{cases} x = x(t) = t + 1 \\ y = y(t) = t^2 - 2t. \end{cases}$$

These equations describe a curve in the xy-plane.

(t is called the parameter)

t	$x = t+1$	$y = t^2 - 2t$	Point (x, y)
0	1	0	$(1,0)$
1	2	-1	$(2,-1)$
2	3	0	$(3,0)$
3	4	3	$(4,3)$
4	5	8	$(5,8)$



* To know the shape of the graph, in many cases it is useful to have an equation of y in terms of x. The process to obtain $y = f(x)$ from the parametric equations is called elimination of parameter.

$$\begin{cases} x = t + 1 \\ y = t^2 - 2t \end{cases}$$

Idea: Solve for t in one equation in terms of x or y. Then plug in the second equation

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$$\begin{cases} x = t + 1 \longrightarrow \boxed{t = x - 1} \\ y = t^2 - 2t \longrightarrow y = (x-1)^2 - 2(x-1) \end{cases}$$

$$y = x^2 - 2x + 1 - 2x + 2$$

$$y = \overbrace{x^2 - 4x + 3}^{f(x)} \longrightarrow \text{curve is a parabola}$$

$a = 1 > 0 \to$ parabola points upward.

Vertex:
$$\left.\begin{aligned} x\text{-vertex} &:= -\frac{b}{2a} = 2 \\ y\text{-vertex} &:= f(2) = -1 \end{aligned}\right\} \text{Vertex } (2, -1)$$

E.g. Given the parametric equations:

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases} \quad 0 \le t < 2\pi.$$

Q: Use elimination of parameter to identify this curve.

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

trig identity

$$\longrightarrow \boxed{x^2 + y^2 = 1} \longrightarrow \text{unit circle.}$$

Calculus of Parametric Curves

Tangent Line Problem:

If a curve is given by the equation $y = f(x)$, how do we find the tangent line to the curve at the point where $x = a$?

$f'(a) \longrightarrow$ Slope of tangent line at $x = a$.

Equation of tangent line: $\boxed{y - f(a) = f'(a)(x - a)}$

Now, the curve is given by

$$\left\{ \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right. \Bigg\| \longrightarrow \text{Find tangent line at a point.}$$

E.g. Given the parametric curve:

$$\begin{cases} x = x(t) = t^2 - 4t \\ y = y(t) = 2t^3 - 6t \end{cases} \quad ; -2 \leq t \leq 3.$$

Q: Find the equation of the tangent line to this curve at the point where $t = 1$.

When $t = 1$: $x = -3$; $y = -4$ $\longrightarrow$ Point $(-3, -4)$

Slope?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{6t^2 - 6}{2t - 4}$$

$$\text{Slope at } \underbrace{(-3, -4)}_{t=1} = \frac{y'(1)}{x'(1)} = \frac{6-6}{2-4} = \frac{0}{-2} = 0$$

Equation of the tangent line at $(-3, -4)$: $\boxed{y = -4}$

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In general, given a parametric curve

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Then:

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}}$$

$$\boxed{\frac{dx}{dy} = \frac{dx/dt}{dy/dt} = \frac{x'(t)}{y'(t)}}$$

Ex. Given $\begin{cases} x = t^3 - 3t \\ y = 3t^2 - 9. \end{cases}$

Find all the x and y-coordinates of the points on this curve at which the tangent line is horizontal

and ____________________ vertical.

$$\frac{dy}{dx} = \frac{6t}{3t^2 - 3}$$

Horizontal tangent line: $\frac{dy}{dx} = 0 \rightarrow \frac{6t}{3t^2-3} = 0$

$\rightarrow 6t = 0 \rightarrow t = 0$

$\rightarrow$ Point $(x(0), y(0)) = \boxed{(0,-9)}$

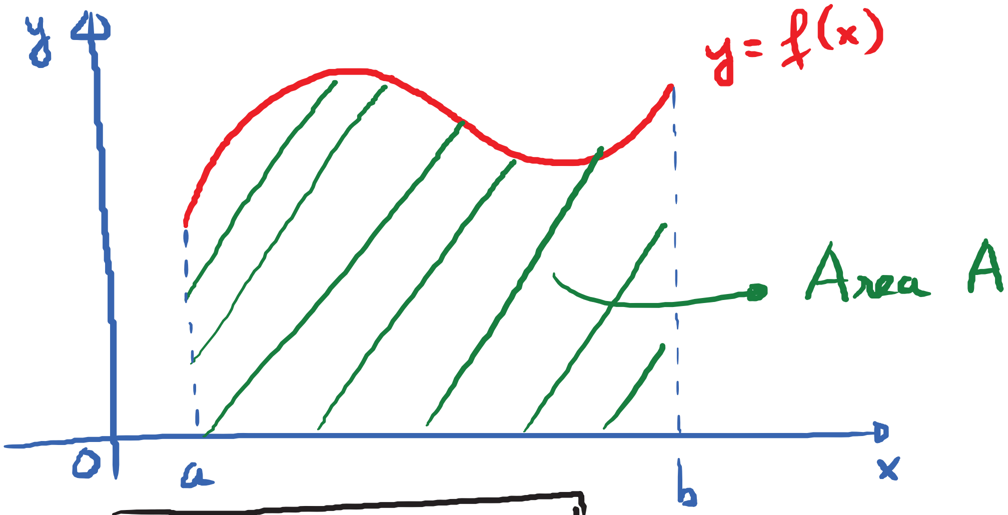
Vertical tangent line: Slope = undefined $\rightarrow \frac{dy}{dx}$ is undefined.

$\rightarrow 3t^2 - 3 = 0 \rightarrow t = \pm 1.$

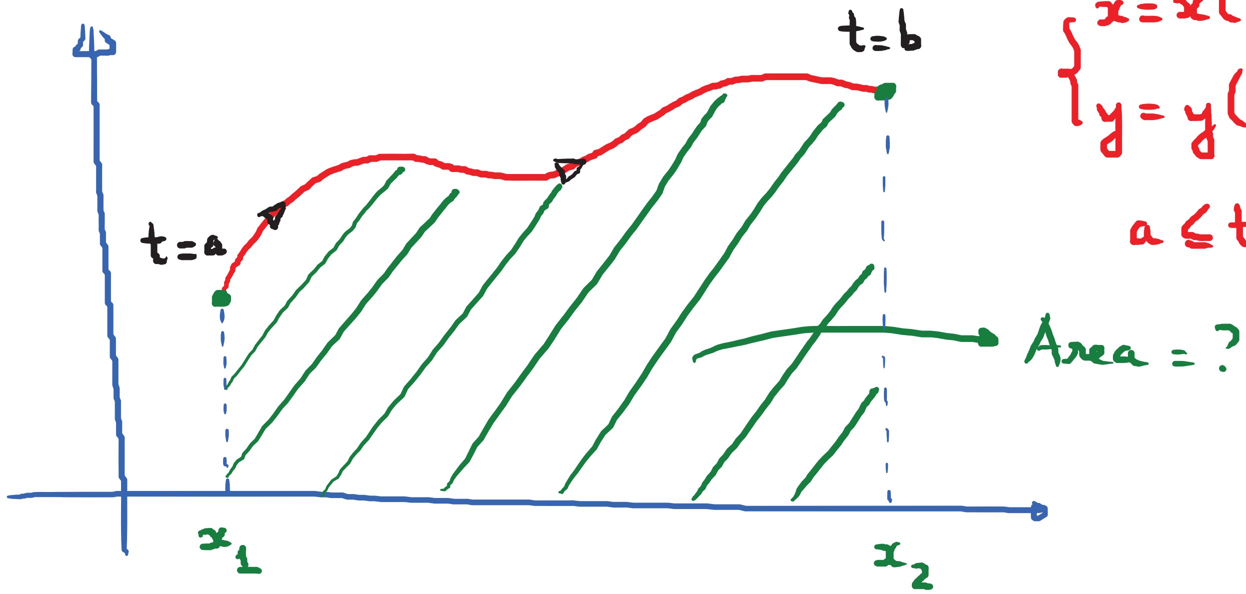
Points: $(x(1), y(1))$; $(x(-1), y(-1))$

$$\boxed{(-2,-6)\,;\,(2,-6)}$$

Area under a parametric curve



$$\boxed{A = \int_a^b f(x)\,dx}$$



$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

$a \leq t \leq b$