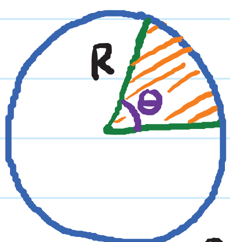
Tangent line : Slope = -1

Point : $x = R\cos\theta = \left(1 + \frac{\sqrt{3}}{2}\right)\cdot\frac{1}{2} = \frac{2+\sqrt{3}}{4}$

$y = R\sin\theta = \left(1 + \frac{\sqrt{3}}{2}\right)\cdot\frac{\sqrt{3}}{2} = \frac{2\sqrt{3}+3}{4}$

Equation: $\boxed{y - \left(\frac{2\sqrt{3}+3}{4}\right) = -1\cdot\left(x - \frac{2+\sqrt{3}}{4}\right)}$

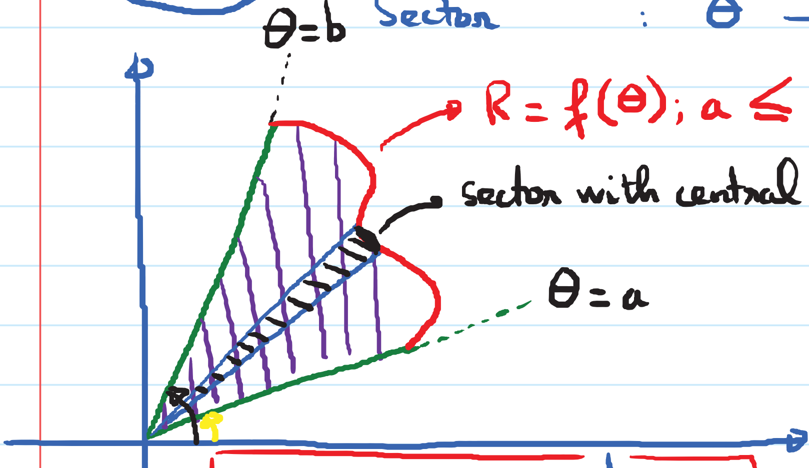
* Areas in Polar Coordinates



$\boxed{\text{Area of sector} = \frac{1}{2}R^2\cdot\theta}$

Whole circle : $2\pi \longrightarrow \pi R^2$

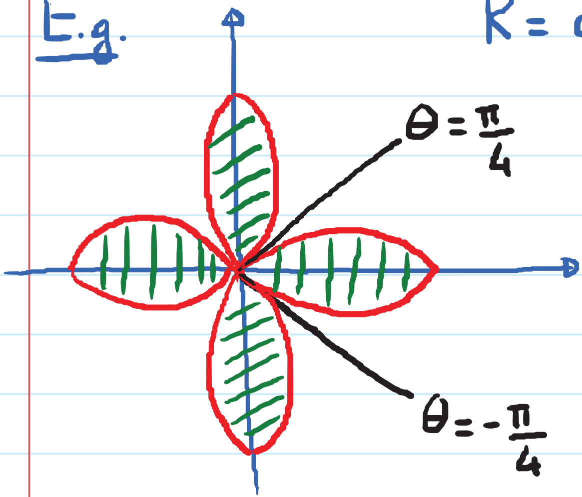
Sector : $\theta \longrightarrow ?$



$A_{sector} = \frac{1}{2}R^2\cdot d\theta$

$$\boxed{\text{Shaded Area} = \frac{1}{2}\int_a^b R^2\,d\theta}$$

E.g. $R = \cos(2\theta)$; $0 \le \theta \le 2\pi$

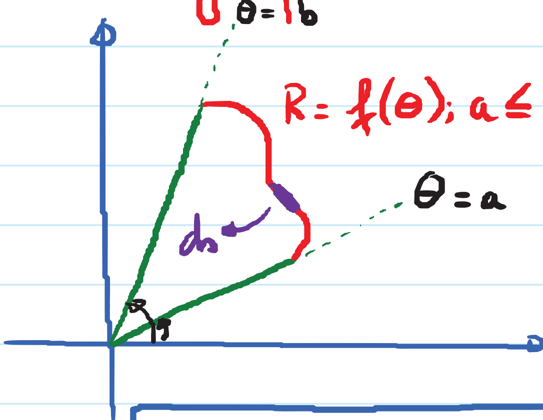


Area of 1 petal:

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} R^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \underbrace{\cos^2(2\theta)}_{\frac{1+\cos(4\theta)}{2}}\, d\theta = \frac{\pi}{8}$$

Shaded area $= 4 \cdot \frac{\pi}{8} = \boxed{\frac{\pi}{2}}$

* Arc length problem



$R = f(\theta)$; $a \le \theta \le b$. Length of this curve?

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$ds = \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right] \cdot (d\theta)^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{\underbrace{\left(\frac{dR}{d\theta}\cos\theta - R\sin\theta\right)}_{\frac{dx}{d\theta}}^2 + \underbrace{\left(\frac{dR}{d\theta}\sin\theta + R\cos\theta\right)}_{\frac{dy}{d\theta}}^2}\; d\theta$$

$$\boxed{ds = \sqrt{\left(\frac{dR}{d\theta}\right)^2 + R^2}\cdot d\theta}$$ arc length element in polar coordinates

So
$$\boxed{L = \int_a^b \sqrt{\left(\frac{dR}{d\theta}\right)^2 + R^2}\, d\theta}$$

E.g. Compute the length of the given curve

(1) $R = \cos\theta$; $0 \le \theta \le \frac{\pi}{2}$; (2) $R = \theta^2$; $0 \le \theta \le \frac{3\pi}{2}$

(3) $R = 9 + 9\sin\theta$; $0 \le \theta \le 2\pi$

Sol: (2) $\displaystyle\int_0^{3\pi/2} \sqrt{(2\theta)^2 + \theta^4}\, d\theta = \int_0^{3\pi/2} \sqrt{4\theta^2 + \theta^4}\, d\theta$

$$= \int_0^{3\pi/2} \theta\sqrt{4 + \theta^2}\, d\theta \;;\; u = 4 + \theta^2 ; \ldots$$

(3) $\displaystyle\int_0^{2\pi} \sqrt{(9\cos\theta)^2 + 81(1 + 2\sin\theta + \sin^2\theta)}\, d\theta$

$$= \int_0^{2\pi} \sqrt{81 + 81(1 + 2\sin\theta)}\, d\theta = 9\cdot \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

$$= 9\sqrt{2} \int_0^{2\pi} \sqrt{1 + \sin\theta} \, d\theta$$

$$= 9\sqrt{2} \cdot \int_0^{2\pi} \sqrt{1 + \sin\theta} \cdot \frac{\sqrt{1 - \sin\theta}}{\sqrt{1 - \sin\theta}} \, d\theta$$

$$= 9\sqrt{2} \cdot \int_0^{2\pi} \frac{\sqrt{1 - \sin^2\theta}}{\sqrt{1 - \sin\theta}} \, d\theta = 9\sqrt{2} \cdot \int_0^{2\pi} \frac{|\cos\theta|}{\sqrt{1 - \sin\theta}} \, d\theta$$

$$= 9\sqrt{2} \left[\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1 - \sin\theta}} \, d\theta - \int_{\pi/2}^{3\pi/2} \frac{\cos\theta}{\sqrt{1 - \sin\theta}} + \int_{3\pi/2}^{2\pi} \frac{\cos\theta}{\sqrt{1 - \sin\theta}} \, d\theta \right.$$

$$- \int \frac{-\cos\theta}{\sqrt{1 - \sin\theta}} \, d\theta \ ; \quad u = 1 - \sin\theta \ ; \ du = -\cos\theta \, d\theta$$

$$- \int \frac{du}{\sqrt{u}} = - \int u^{-1/2} \, du = - \frac{u^{1/2}}{1/2} = -2u^{1/2}$$

$$= -2\sqrt{1 - \sin\theta}$$