Zeros of Polynomial Functions.

Monday, March 25, 2019 1:00 PM

Rational Zero Theorem:

If $\frac{p}{q}$ is a rational zero of a polynomial functions with integer coefficients, then p must be a factor of the constant term and q must be a factor of the leading coefficient.

E.g. $f(x) = 4x^3 + 8x^2 + x - 3$

List of possible rational zeros:

$$\frac{p}{q} = \frac{\text{factor of } -3}{\text{factor of } 4} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2, \pm 4}$$

$$= \left\{ \pm 1; \pm \frac{1}{2}; \pm \frac{1}{4}, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4} \right\}$$

→ Apply The Rational Zero Theorem to find zeros (roots) of polynomials.

E.g. $f(x) = 2x^3 - 11x^2 + 8x + 7$.

Q: Find the zeros of f.

Step 1: Apply the Rational Zero Theorem to make a list of possible rational zeros of f.

$$\frac{p}{q} = \frac{p \text{ is a factor } 7}{q \text{ is a factor } 2} = \frac{\pm 1, \pm 7}{\pm 1, \pm 2}$$

$$\text{List} = \left\{ \pm 1, \pm \frac{1}{2}, \pm 7, \pm \frac{7}{2} \right\}$$

Step 2: Test numbers in the list until we find a zero. Use synthetic division to test.

* Test $c = -7$

$$\begin{array}{r|rrrr} -7 & 2 & -11 & 8 & 7 \\ & & -14 & 175 & \\ \hline & 2 & -25 & 183 & \times \end{array}$$

Remainder $\neq 0$
Not a root.

* Test $c = -\frac{1}{2}$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -11 & 8 & 7 \\ & & -1 & 6 & -7 \\ \hline & \boxed{2 \quad -12 \quad 14} & & & \boxed{0} \end{array}$$

$q(x) = 2x^2 - 12x + 14$

Remainder $= 0$

So, we found the first zero; that is, $\boxed{x = -\frac{1}{2}}$

Step 3: Find the remaining zeros by setting the quotient $q(x) = 0$ and solve for x

$q(x) = 2x^2 - 12x + 14 = 0 \rightarrow$ Solve for x

$\rightarrow x^2 - 6x + 7 = 0$

$\rightarrow$ quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{6 \pm \sqrt{36 - 28}}{2} = \frac{6 \pm \sqrt{8}}{2}$$

$\sqrt{8} \rightarrow \sqrt{4} \cdot \sqrt{2}$

$$x = \frac{6 \pm 2\sqrt{2}}{2} = \boxed{3 \pm \sqrt{2}.} \rightarrow \text{other zeros}$$

Conclusion: Zeros of f: $\boxed{\left\{-\frac{1}{2},\ 3 + \sqrt{2},\ 3 - \sqrt{2}\right\}}$

Ex. Find all the zeros of the polynomial

$$g(x) = 3x^4 + 2x^3 + 2x^2 + 2x - 1.$$

Step 1:

$$\frac{p}{q} = \frac{\text{factor of } -1}{\text{factor of } 3} = \frac{\pm 1}{\pm 1, \pm 3}$$

$$\text{List} = \left\{ \pm 1, \pm \frac{1}{3} \right\}$$

Step 2:

$$\begin{array}{r|rrrrr} -1 & 3 & 2 & 2 & 2 & -1 \\ & & -3 & 1 & -3 & 1 \\ \hline & 3 & -1 & 3 & -1 & \boxed{0} \end{array} \rightarrow R = 0$$

$\rightarrow$ -1 is a zero

$$q(x) = 3x^3 - x^2 + 3x - 1$$

Step 3: Set $q(x) = 0$

$$\underbrace{3x^3 - x^2} + 3x - 1 = 0$$

$\rightarrow$ $x^2(3x-1) + (3x-1) = 0$

$\rightarrow$ $(3x-1)(x^2+1) = 0$

$\rightarrow$ $3x - 1 = 0$ or $x^2 + 1 = 0$

$\rightarrow$ $x = \frac{1}{3}$ $\quad\rightarrow$ $x^2 = -1$

$\rightarrow$ $x = \pm\sqrt{-1} = \pm i$

Zeros of q are: $\boxed{\left\{-1, \frac{1}{3}, i, -i\right\}}$

linear factors of a polynomial.

If $c_1, c_2, \ldots, c_n$ are zeros of a polynomial f of degree n, then f can be written as:

$$\boxed{f(x) = a(x-c_1)(x-c_2)\cdots(x-c_n)}$$

(here a is a constant, the leading coeff. of f).

<u>E.g.</u> f is a polynomial of degree 3.

$-1, 2, 3$ are the zeros of f.

$\boxed{f(0) = 12.}$

Find $f(x)$.

<u>Sol:</u> $f(x) = a\cdot(x+1)(x-2)(x-3)$

Find the leading coeff. a.

$$12 = f(0) = a\cdot\boxed{(0+1)(0-2)(0-3)}$$

$$12 = 6a \longrightarrow a = 2.$$