6.1 - Inverse Trig Functions.

Monday, April 15, 2019 10:18 AM

Inverse Sine Function.

The inverse sine function $\underbrace{y = \sin^{-1}(x)}_{\text{read as inverse sine of } x}$ or $y = \underbrace{\arcsin}_{\text{read as arc sine of } x}(x)$

$\sin^{-1}(x)$ gives us the angle y whose sine is equal to x.

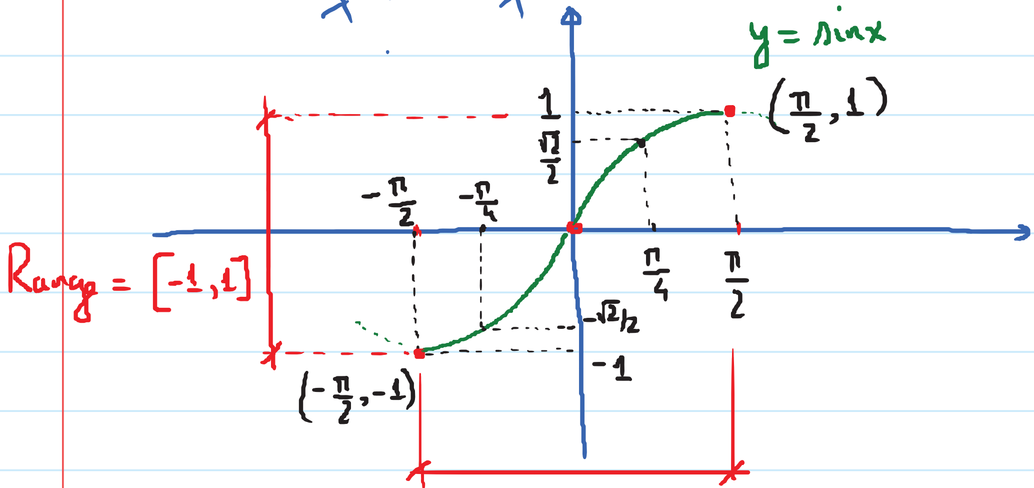
In short, $y = \sin^{-1}(x)$ is equivalent to $x = \sin(y)$

E.g. $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ because $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

E.g.

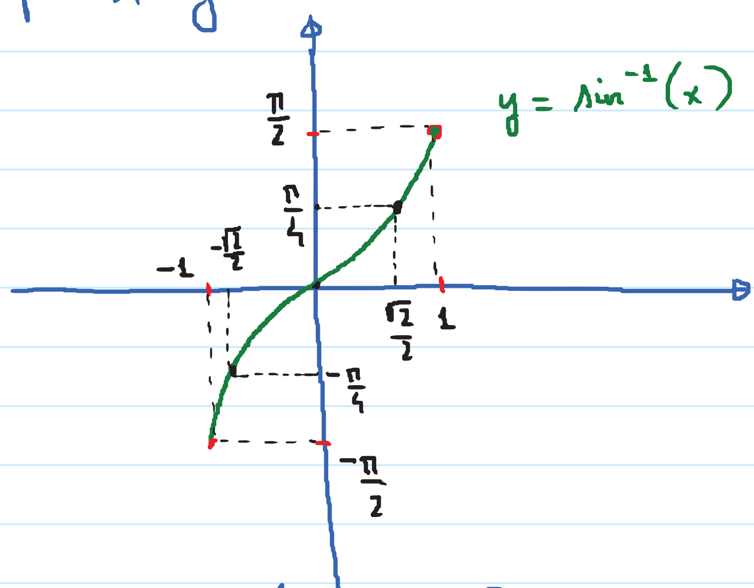
Angle	Sine of Angle	#	Inverse Sine of #
0	0	0	$\sin^{-1}(0) = 0$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$
$\frac{\pi}{2}$	1	1	$\sin^{-1}(1) = \frac{\pi}{2}$

Restriction of sine function



Domain : Restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

→ Graph of $y = \sin^{-1}(x)$



Domain of $y = \sin^{-1}(x)$ is $[-1, 1]$

Range of $y = \sin^{-1}(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ — 1st or 4th quadrant

Note: $y = \sin^{-1}(x)$ gives us angles in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ only

E.g. Find the exact value w/o a calculator:

(a) $\sin^{-1}(-1)$ (b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

(c) $\sin^{-1}\left(-\frac{1}{2}\right)$ (d) $\sin^{-1}(2)$

Sol: (a) $\sin^{-1}(-1)$ = angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is -1

$= \boxed{-\frac{\pi}{2}}$

(b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ = angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{\sqrt{2}}{2}$.

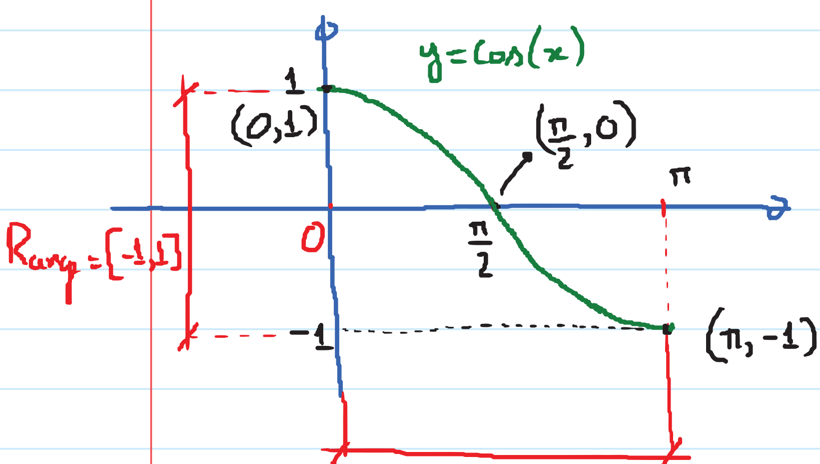
$= \boxed{-\frac{\pi}{4}}$

(c) $\sin^{-1}\left(-\frac{1}{2}\right)$ = angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{1}{2}$

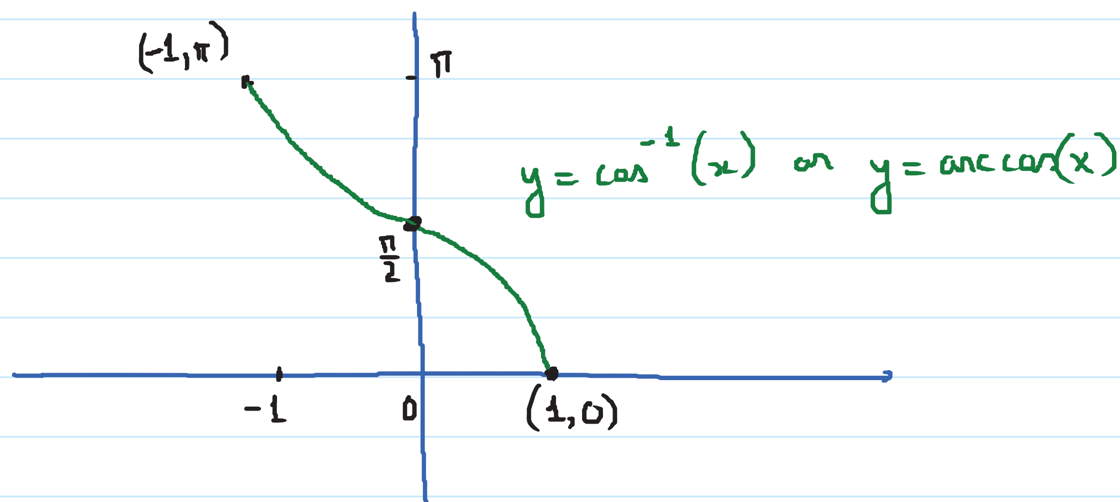
$= \boxed{-\frac{\pi}{6}}$

(d) $\sin^{-1}(2)$ = angle whose sine is 2
= DNE (doesn't exist)

Inverse Cosine.



Domain: Restricted to [0,π]



Graph of the inverse cosine.

Domain of $y = \cos^{-1}(x)$: $[-1,1]$

Range of $y = \cos^{-1}(x)$: $[0,\pi]$.

The function $y = \cos^{-1}(x)$ or $y = \arccos(x)$ gives us the angle y in $\underbrace{[0, \pi]}_{\text{QI or QII}}$ whose cosine is equal to x.

In short, $y = \cos^{-1}(x)$ is equivalent to $\cos(y) = x$.

E.g.

a) $\cos^{-1}\left(\frac{1}{2}\right)$ = angle in $[0, \pi]$ whose cosine $= \frac{1}{2}$.

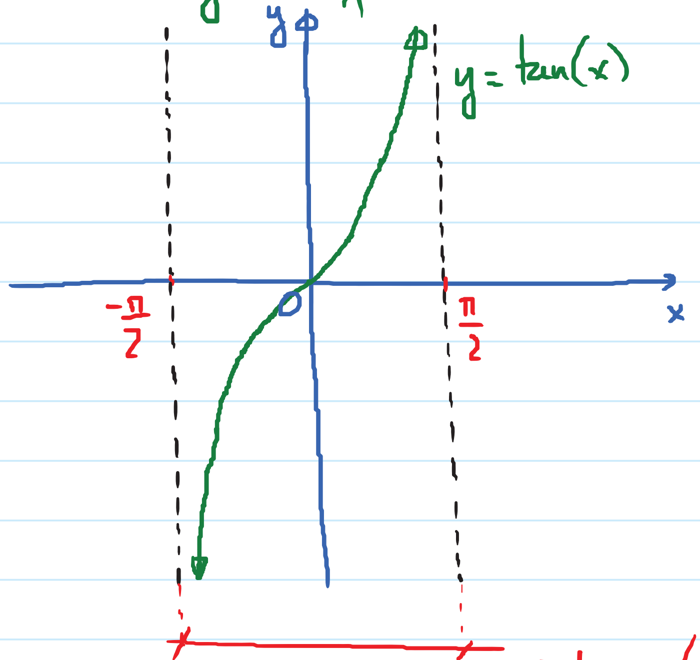
$= \boxed{\frac{\pi}{3}}$

b) $\cos^{-1}\left(-\frac{1}{2}\right)$ = angle in $[0, \pi]$ whose cosine $= -\frac{1}{2}$

$= \boxed{\frac{2\pi}{3}}$

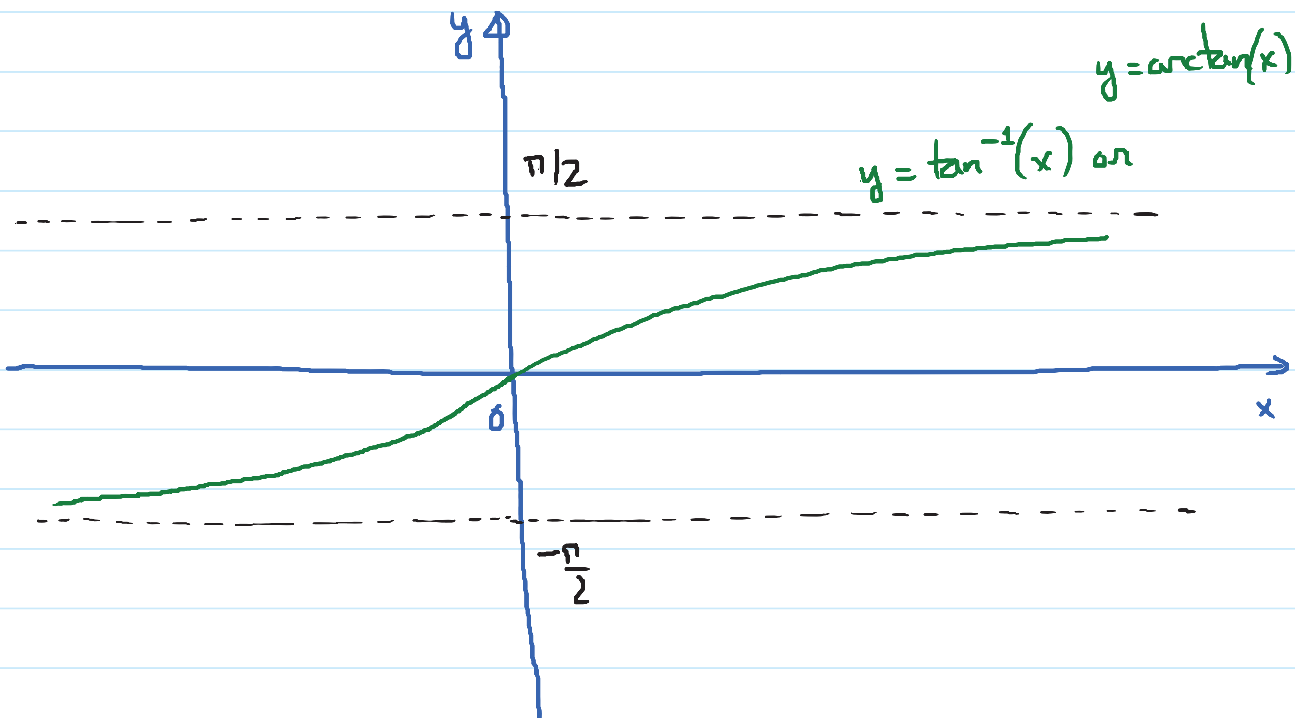
c) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$

Inverse Tangent function.



Range = $(-\infty, \infty)$

Domain is restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$y = \arctan(x)$

$y = \tan^{-1}(x)$ or

Domain = $(-\infty, \infty)$

Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = \tan^{-1}(x)$ or $y = \arctan(x)$ gives us the angle y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is equal to x.

(QI or QIV)

In short, $y = \tan^{-1}(x)$ is equivalent to $\tan(y) = x$.

E.g. Find $\tan^{-1}(1) = \frac{\pi}{4}$

Find $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

Algebraic Calculations with Inverse Trig Functions.

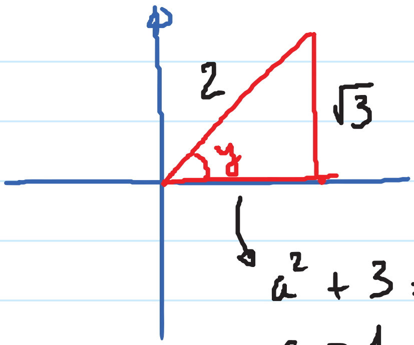
E.g. Find the exact value of $\tan\left(\boxed{\arcsin\left(\frac{\sqrt{3}}{2}\right)}\right)$

1st method: calculator:

$$\frac{\sqrt{3}}{2} \longrightarrow \sin^{-1} \longrightarrow 60^\circ \longrightarrow \tan \longrightarrow \sqrt{3}$$

2nd method: by hand.

Let $y = \arcsin\left(\frac{\sqrt{3}}{2}\right)$. Then: $\sin(y) = \frac{\sqrt{3}}{2}$ and y is in QI.



$a^2 + 3 = 4$

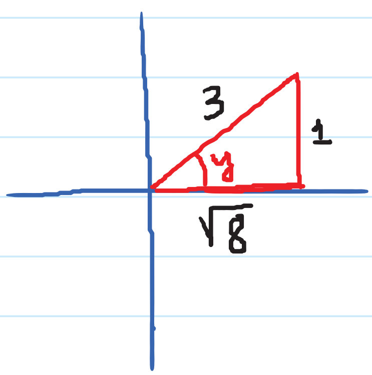
$a = 1$

We need $\tan(y)$.

$\tan(y) = \frac{\sqrt{3}}{1} = \sqrt{3}$.

Ex. Find the exact value of $\tan\Big(\underbrace{\arcsin\left(\tfrac{1}{3}\right)}_{y}\Big)$

Let $y = \arcsin\left(\frac{1}{3}\right)$. Then $\sin y = \frac{1}{3}$; y is in QI



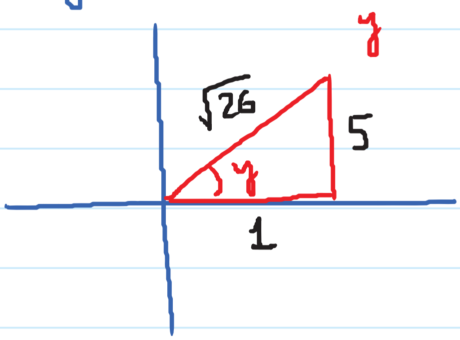
Need: $\tan(y) = \frac{\text{opp.}}{\text{adj.}}$

So, $\tan(y) = \frac{1}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{8}}$

$= \frac{\sqrt{8}}{8} = \frac{2\sqrt{2}}{8}$

$= \boxed{\frac{\sqrt{2}}{4}}$

E.g. $\cos\Big(\underbrace{\tan^{-1}(5)}_{y}\Big)$

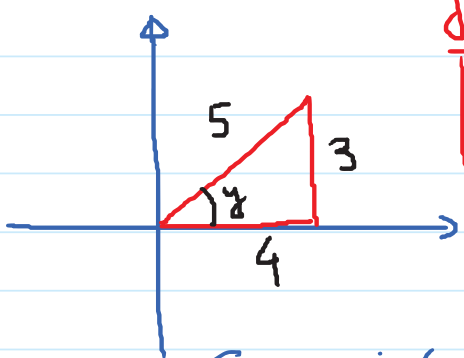


$\cos(y) = \frac{1}{\sqrt{26}} = \frac{\sqrt{26}}{26}$.

$$\sin\left(2\underbrace{\tan^{-1}\left(\tfrac{3}{4}\right)}_{y}\right) = \sin(2y)$$

$$= 2\boxed{\sin(y)\cos(y)}$$

Double-Angle Identity



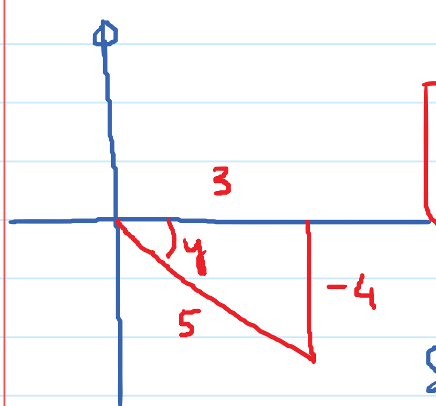
$$\boxed{\sin(y) = \frac{3}{5}\ ;\ \cos(y) = \frac{4}{5}}.$$

So, $\sin(2y) = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \boxed{\frac{24}{25}}$

E.g. $\cos\left(2\underbrace{\arctan\left(-\frac{4}{3}\right)}_{y}\right) = \cos(2y)$

$$= \cos^2(y) - \sin^2(y)$$

Double-angle for cosine

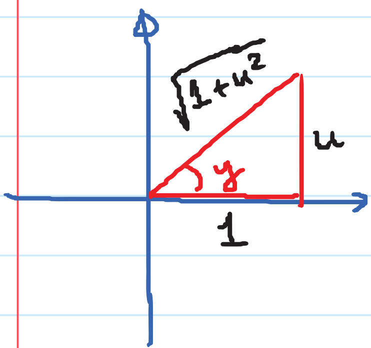


$$\boxed{\cos(y) = \frac{3}{5}\ ;\ \sin(y) = -\frac{4}{5}}$$

So, $\cos(2y) = \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = \frac{9-16}{25}$

$$= \boxed{-\frac{7}{25}}$$

E.g. $\sin(\underbrace{\arctan(u)}_{y})$ $\tan(y) = u$



Want: $\sin(y) = \dfrac{u}{\sqrt{1+u^2}} \cdot \dfrac{\sqrt{1+u^2}}{\sqrt{1+u^2}}$

$$\boxed{\sin(y) = \frac{u\sqrt{1+u^2}}{1+u^2}}$$