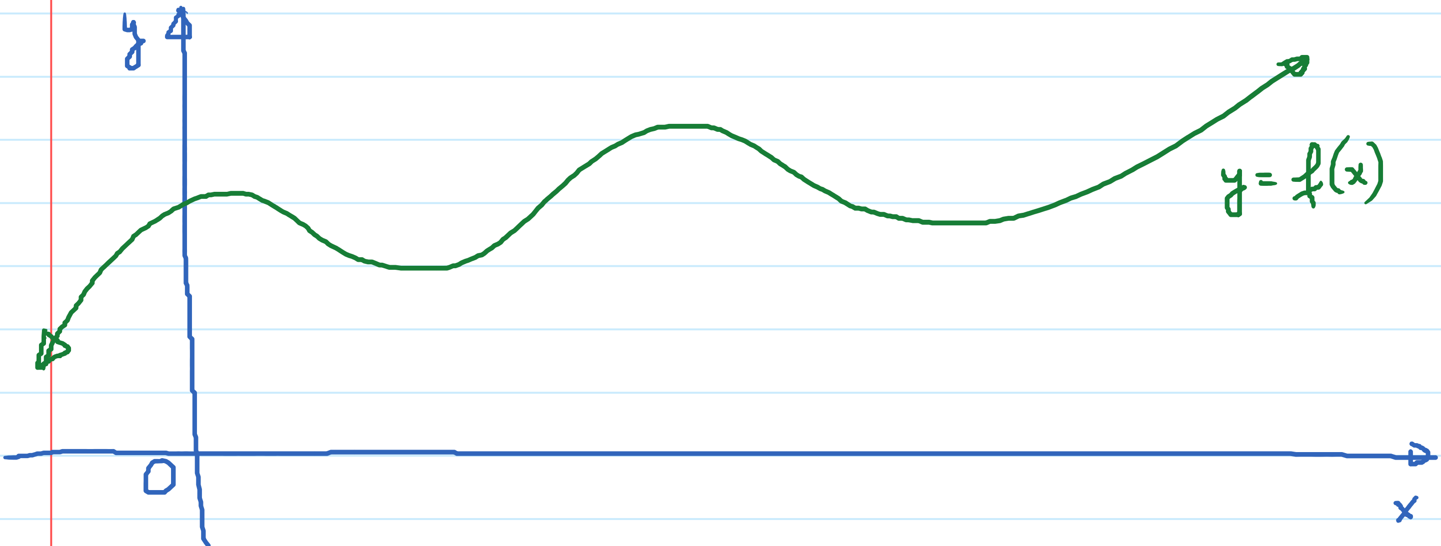
2.4. Continuity

Wednesday, January 23, 2019 8:06 AM

Goals: (1) The "limit" definition of continuity.

(2) Classify different types of "discontinuity"

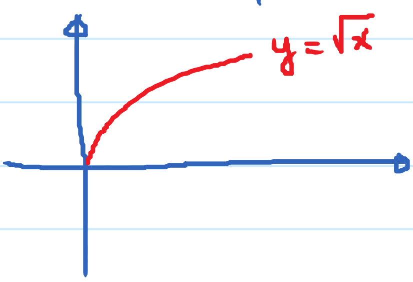
Intuitive concept of continuity.



Definition of continuity at a point

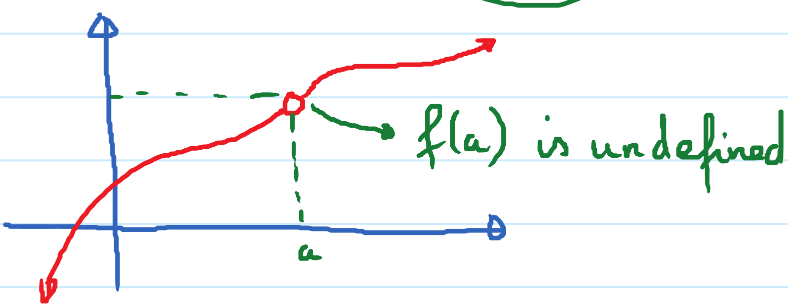
Analyze situations where a function $y = f(x)$ fails to be continuous at a point $x = a$.

* Point $x = a$ must be in the domain of f; i.e., $f(a)$ must be defined.



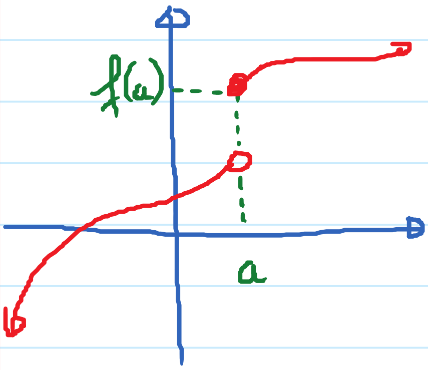
It is NOT continuous at $x = -5$

f is not defined here



* $2^{\underline{nd}}$ requirement for f to be continuous at $x=a$ is

$$\lim_{x\to a} f(x) \text{ exists } \left(\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \right)$$

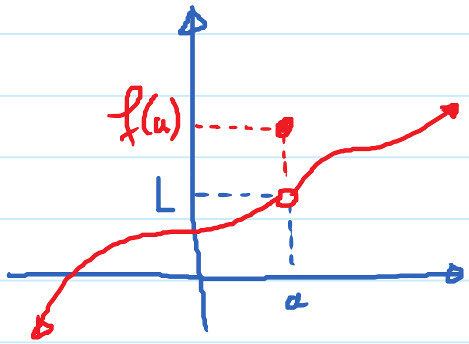


E.g where $\lim_{x\to a} f(x)$ DNE.

(Not continuous at $x=a$)

* $3^{\underline{rd}}$ requirement for f to be continuous at $x=a$ is

$$f(a) = \lim_{x\to a} f(x)$$



E.g. where $f(a) \neq \lim_{x\to a} f(x)$

(Not continuous at $x=a$)

__Definition:__ A function $y=f(x)$ is continuous at a point $x=a$ if all of the following conditions are satisfied:

1. $f(a)$ must be defined.
2. $\lim_{x\to a} f(x)$ must exist.
3. $f(a) = \lim_{x\to a} f(x)$

E.g. $f(x) = \dfrac{1}{4x+3}$

f is NOT continuous at $x = -\frac{3}{4}$ b/c $f(-\frac{3}{4})$ is undefined (f violates 1^{st} requirement at $x = -\frac{3}{4}$)

E.g.

$$f(x) = \begin{cases} x+2 & \text{if } x > 3 \\ x-2 & \text{if } x \leq 3 \end{cases}$$

Consider $x = 3$.

$$\left.\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3}(x-2) = 1 \\ \lim_{x\to 3^+} f(x) &= \lim_{x\to 3}(x+2) = 5 \end{aligned}\right\} 1 \neq 5$$

$\rightarrow \lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x) \rightarrow \lim_{x\to 3} f$ DNE.

So, f is NOT continuous at $x = 3$ b/c it violates 2^{nd} requirement at $x = 3$.

E.g. Consider

$$f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$$

Q: Use the definition of continuity to verify that this function is continuous at the point $x = 0$

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(1) Is $f(0)$ defined? Yes, $f(0) = -1$

(2) Does $\lim_{x \to 0} f(x)$ exist? Yes.

$$\left.\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0} (x^2 - e^x) = -1 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0} (x - 1) = -1 \end{aligned}\right\}$$

$\longrightarrow$ left limit = Right limit

$\longrightarrow$ $\lim_{x \to 0} f(x)$ exists and $\lim_{x \to 0} f(x) = -1$.

(3) Is $f(0) = \lim_{x \to 0} f(x)$? Yes, they are both -1

E.g. Consider

$$g(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 1} & \text{if } x < 1 \\ 5 & \text{if } x \geq 1 \end{cases}$$

Is the function g continuous at $x = 1$? Explain why?

No. It violates 2^{nd} requirement.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x-2)}{\cancel{x-1}}$$

$$= \lim_{x \to 1} (x - 2) = -1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (5) = 5$$

So, left limit $\neq$ Right limit. So, $\lim_{x \to 1} f(x)$ DNE.

So, f is NOT continuous at $x = 1$.

E.g. Consider

$$h(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 2019 & \text{if } x = 0 \end{cases}$$

Q: Is h continuous at $x = 0$? Explain why?

1. Is $h(0)$ defined? Yes, $h(0) = 2019$

2. Does $\lim_{x \to 0} h(x)$ exist? Yes.

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

3. Is $h(0) = \lim_{x \to 0} h(x)$? NO.

Conclusion: h is NOT continuous at $x = 0$.

* Note: We say that a function f is continuous on an interval (c, d) if f is continuous at every single point within that interval.