3.6 The Chain Rule

Wednesday, February 13, 2019 9:48 AM

Function $\boxed{y = f(x)}$	Derivative $\frac{dy}{dx}$
$y = x^2$	$\frac{d}{dx}(x^2) = 2x$
f is the square function	
$y = x^n$	$\frac{d}{dx}[x^n] = nx^{n-1}$
$y = \frac{1}{x}$	$\frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2}$
$y = \sqrt{x}$	$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$
$y = \sin x$	$\frac{d}{dx}[\sin x] = \cos x$
$y = \cos x$	$\frac{d}{dx}[\cos x] = -\sin x$
$y = \tan x$	$\frac{d}{dx}[\tan x] = \sec^2 x$
$y = \sec x$	$\frac{d}{dx}[\sec x] = \sec x \tan x$

What if we have the following functions

Function $y = \boxed{f}(\underbrace{g(x)}_{\text{inside}})$ outside	Derivative $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$
$(\underbrace{\boxed{\sin x + 5x}}_{g(x)})^{\boxed{2}}$ f: square function	$\underbrace{2(\sin x + 5x)}_{f'(g(x))} \cdot \underbrace{(\cos x + 5)}_{g'(x)}$
$(\boxed{6x^3 - 5x^2 + 7})^{\boxed{n}}$	$n(6x^3 - 5x^2 + 7)^{n-1} \cdot (18x^2 - 10x)$
$\frac{1}{\boxed{\cos x}}$	$-\frac{1}{\cos^2 x} \cdot (-\sin x) = \frac{\sin x}{\cos^2 x}$
$\sqrt{\boxed{3x^2 + 7}}$	$\frac{1}{2\sqrt{3x^2+7}} \cdot 6x = \frac{3x}{\sqrt{3x^2+7}}$
$\sin(\boxed{\cos x})$	
$\cos(\boxed{3x^2 - 5})$	
$\tan(\boxed{x + \sqrt{x}})$	
$\sec(\boxed{x - \frac{1}{x}})$	

derivative of inside

Chain Rule: $[f(g(x))]' = \boxed{f'(g(x))} \cdot \boxed{g'(x)}$

derivative of outside evaluated at inside

Chain Rule using Leibnitz notation

E.g. $y = \sin(\underbrace{\sqrt{x}}_{u})$. Find $\dfrac{dy}{dx}$

Let $\boxed{u = \sqrt{x}.}$ Then $\boxed{y = \sin(u)}$

y is a function of u. u is a function of x.

$$\begin{tikzcd} y \arrow[r, dash] \arrow[r, dash, bend left, "dy/du"] \arrow[rr, dash, bend right] & u \arrow[r, dash] \arrow[r, dash, bend left, "du/dx"] & x \end{tikzcd}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$y = \sin(u) \longrightarrow \dfrac{dy}{du} = \cos u.$

$u = \sqrt{x} \longrightarrow \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$

So, $\dfrac{dy}{dx} = \cos u \cdot \dfrac{1}{2\sqrt{x}}$

$$\boxed{\frac{dy}{dx} = \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}}$$

E.g. $y = \left(\boxed{\cos x + \csc x}\right)^{10}$ Find $\dfrac{dy}{dx}$.

Let $u = \cos x + \csc x$. Then $y = u^{10}$

$$y \text{ —— } u \text{ —— } x$$

$$y = u^{10} \longrightarrow \frac{dy}{du} = 10u^9$$

$$u = \cos x + \csc x \longrightarrow \frac{du}{dx} = -\sin x - \csc x \cot x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 10u^9 \cdot \left(-\sin x - \csc x \cdot \cot x\right)$$

$$\frac{dy}{dx} = 10\left(\cos x + \csc x\right)^9 \cdot \left(-\sin x - \csc x \cdot \cot x\right)$$

Chain Rule with more than 2 components.

$$y \overset{dy/dw}{\text{ —— }} w \overset{dw/du}{\text{ —— }} u \overset{du/dx}{\text{ —— }} x$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}}$$

E.g. $y = \sin\left(\overset{b}{\boxed{\cos}}\left(\overset{a}{\boxed{\frac{1}{x}}}\right)\right)$. Find $\frac{dy}{dx}$

Let $a = \frac{1}{x}$. Let $b = \cos(a)$. $y = \sin(b)$

$$y \;\text{———}\; b \;\text{———}\; a \;\text{———}\; x$$

$$\frac{dy}{dx} = \boxed{\frac{dy}{db}} \cdot \boxed{\frac{db}{da}} \cdot \boxed{\frac{da}{dx}}$$

$y = \sin b \rightarrow \frac{dy}{db} = \boxed{\cos b}$

$b = \cos(a) \rightarrow \frac{db}{da} = \boxed{-\sin a}$

$a = \frac{1}{x} \rightarrow \frac{da}{dx} = \boxed{-\frac{1}{x^2}}$

$$\frac{dy}{dx} = \cos b \cdot (-\sin a) \cdot \left(-\frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = \cos\left(\cos\left(\frac{1}{x}\right)\right) \cdot \left(-\sin\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right)$$

$$= \boxed{\cos\left(\cos\left(\frac{1}{x}\right)\right) \cdot \sin\left(\frac{1}{x}\right) \cdot \frac{1}{x^2}}$$

E.g. $y = \cos\left(\overbrace{\sqrt{\underbrace{\boxed{\sin}}_{c}\left(\underbrace{\boxed{\tan}}_{b}\left(\underbrace{\boxed{\pi x}}_{a}\right)\right)}}^{g}\right)$

Find $\dfrac{dy}{dx}$.

$a = \pi x$. $\quad b = \tan(a)$. $\quad c = \sin(b)$

$g = \sqrt{c}$. $\quad y = \cos(g)$

y — g — c — b — a — x

$$\frac{dy}{dx} = -\sin(g) \cdot \frac{1}{2\sqrt{c}} \cdot \cos b \cdot \sec^2 a \cdot \pi$$

$$\boxed{-\sin\left(\sqrt{\sin(\tan(\pi x))}\right) \cdot \frac{1}{2\sqrt{\sin(\tan(\pi x))}} \cos(\tan(\pi x)) \cdot \sec^2(\pi x) \cdot \pi} = \frac{dy}{dx}$$

Monday, February 18, 2019 9:10 AM

Chain Rule:

Newton's notation: $\left[f(g(x))\right]' = f'(g(x)) \cdot g'(x)$

Leibnitz notation: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ $(y — u — x)$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx} \quad (y — w — u — x)$$