

## 3.6 The Chain Rule

Wednesday, February 13, 2019

9:48 AM

Function  $y = f(x)$

Derivative  $\frac{dy}{dx}$

$y = x^2$   
 $f$  is the square function

$$\frac{d}{dx}(x^2) = 2x$$

$$y = x^n$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$y = \frac{1}{x}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2}$$

$$y = \sqrt{x}$$

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$$

$$y = \sin x$$

$$\frac{d}{dx}[\sin x] = \cos x$$

$$y = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$y = \tan x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$y = \sec x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

What if we have the following functions

Function  $y = f(g(x))$   
 out side inside

Derivative

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$(\sin x + 5x)^2$   $f$ : square function

$$2(\sin x + 5x) \cdot (\cos x + 5)$$

$f'(g(x)) \cdot g'(x)$

$(6x^3 - 5x^2 + 7)^n$

$$n(6x^3 - 5x^2 + 7)^{n-1} \cdot (18x^2 - 10x)$$

$\frac{1}{\cos x}$

$$\rightarrow -\frac{1}{\cos^2 x} \cdot (-\sin x) = \frac{\sin x}{\cos^2 x}$$

$\sqrt{3x^2 + 7}$

$$\rightarrow \frac{1}{2\sqrt{3x^2 + 7}} \cdot 6x = \frac{3x}{\sqrt{3x^2 + 7}}$$

$\sin(\cos x)$

$\cos(3x^2 - 5)$

$\tan(x + \sqrt{x})$

$\sec\left(x - \frac{1}{x}\right)$

Chain Rule:  $[f(g(x))]' = f'(g(x)) \cdot g'(x)$   
 derivative of outside evaluated at inside derivative of inside

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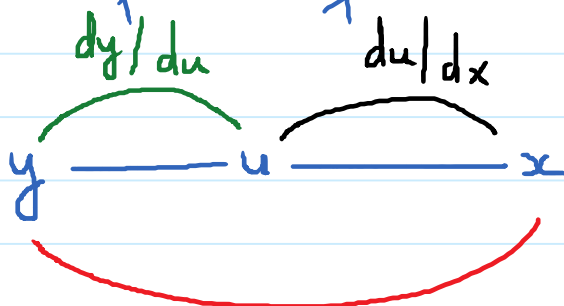
# Chain Rule using Leibnitz notation

E.g.  $y = \sin(\sqrt{x})$ . Find  $\frac{dy}{dx}$

$\underbrace{\hspace{1cm}}_u$

Let  $u = \sqrt{x}$ . Then  $y = \sin(u)$

$y$  is a function of  $u$ .  $u$  is a function of  $x$ .



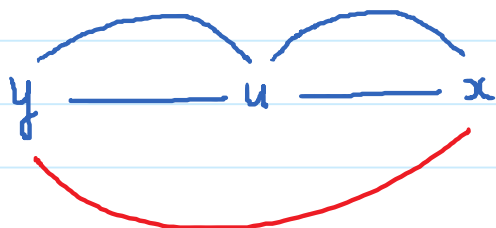
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\left. \begin{array}{l} y = \sin(u) \rightarrow \frac{dy}{du} = \cos u. \\ u = \sqrt{x} \rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \end{array} \right\} \text{ So, } \frac{dy}{dx} = \cos u \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

E.g.  $y = (\cos x + \csc x)^{10}$  Find  $\frac{dy}{dx}$ .

Let  $u = \cos x + \csc x$ . Then  $y = u^{10}$



$$y = u^{10} \rightarrow \frac{dy}{du} = 10u^9$$

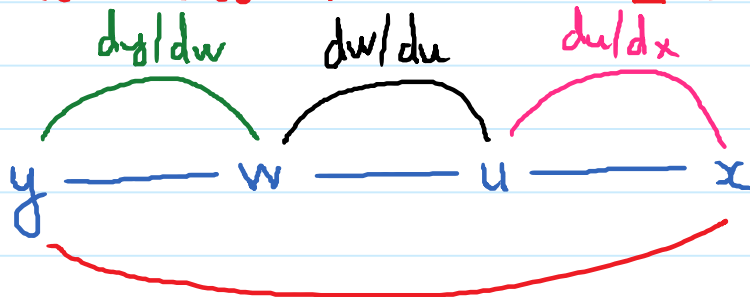
$$u = \cos x + \csc x \rightarrow \frac{du}{dx} = -\sin x - \csc x \cot x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 10u^9 \cdot (-\sin x - \csc x \cot x)$$

$$\frac{dy}{dx} = 10(\cos x + \csc x)^9 \cdot (-\sin x - \csc x \cot x)$$

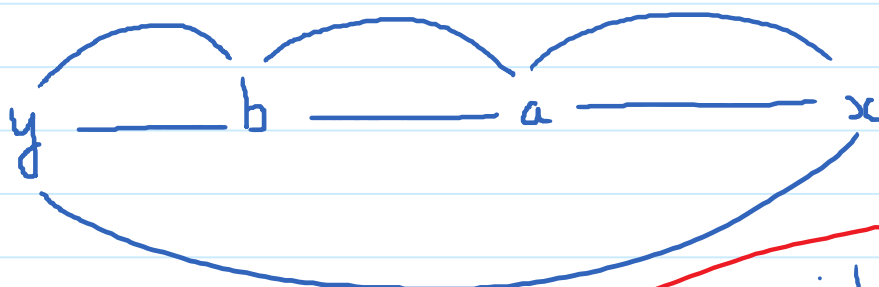
Chain Rule with more than 2 components.



$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

E.g.  $y = \sin \left( \overset{b}{\boxed{\cos}} \left( \overset{a}{\boxed{\frac{1}{x}}} \right) \right)$ . Find  $\frac{dy}{dx}$

Let  $a = \frac{1}{x}$ . Let  $b = \cos(a)$ .  $y = \sin(b)$



$$\frac{dy}{dx} = \boxed{\frac{dy}{db}} \cdot \boxed{\frac{db}{da}} \cdot \boxed{\frac{da}{dx}}$$

$$y = \sin b \rightarrow \frac{dy}{db} = \boxed{\cos b}$$

$$b = \cos(a) \rightarrow \frac{db}{da} = \boxed{-\sin a}$$

$$a = \frac{1}{x} \rightarrow \frac{da}{dx} = \boxed{-\frac{1}{x^2}}$$

$$\frac{dy}{dx} = \cos b \cdot (-\sin a) \cdot \left(-\frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = \cos\left(\cos\left(\frac{1}{x}\right)\right) \cdot \left(-\sin\left(\frac{1}{x}\right)\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$= \boxed{\cos\left(\cos\left(\frac{1}{x}\right)\right) \cdot \sin\left(\frac{1}{x}\right) \cdot \frac{1}{x^2}}$$

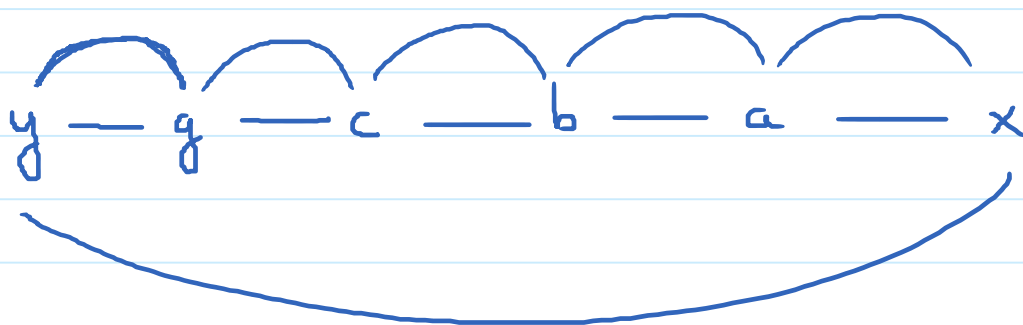
E.g.  $y = \cos(\sqrt{\sin(\tan(\pi x))})$

Diagram showing the chain rule layers:  $\pi x$  (labeled  $a$  in red),  $\tan$  (labeled  $b$  in green),  $\sin$  (labeled  $c$  in black), and  $\sqrt{\phantom{x}}$  (labeled  $g$  in purple).

Find  $\frac{dy}{dx}$ .

$a = \pi x$  .  $b = \tan(a)$  .  $c = \sin(b)$

$g = \sqrt{c}$  .  $y = \cos(g)$



$$\frac{dy}{dx} = -\sin(g) \cdot \frac{1}{2\sqrt{c}} \cdot \cos b \cdot \sec^2 a \cdot \pi$$

$$-\sin(\sqrt{\sin(\tan(\pi x))}) \cdot \frac{1}{2\sqrt{\sin(\tan(\pi x))}} \cos(\tan(\pi x)) \cdot \sec^2(\pi x) \cdot \pi = \frac{dy}{dx}$$

## Chain Rule:

Newton's notation:  $\left[ f(g(x)) \right]' = f'(g(x)) \cdot g'(x)$

Leibnitz notation:  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (y - u - x)$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx} \quad (y - w - u - x)$$