3.7. Derivatives of Inverse Trig Functions.

Monday, February 18, 2019 9:51 AM

(1) Brief Review of Inverse Trig Functions.

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \longrightarrow \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \longrightarrow \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

To sum up,

$\arcsin(x)$ gives you the angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is equal to y.

$$\arcsin(x) = y \longleftrightarrow \sin(y) = x.$$

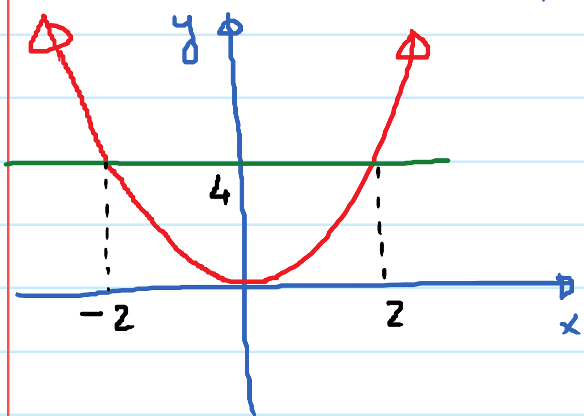
$\arccos(x)$ gives you the angle y in $[0, \pi]$ whose cosine is equal to y

$$\arccos(x) = y \longleftrightarrow \cos(y) = x$$

(HW: Review arctan, arcsec, arccsc, arccot ...)

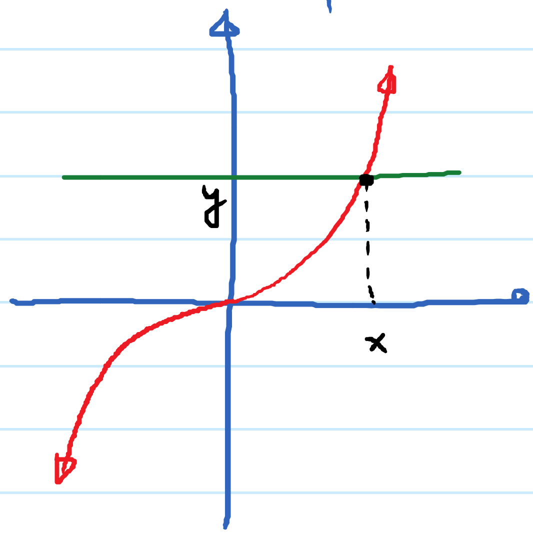
Other notation: $\left.\begin{array}{l} \arcsin(x) \equiv \sin^{-1}(x) \\ \arccos(x) \equiv \cos^{-1}(x) \end{array}\right\}$ inverse functions

In general, if $y = f(x)$ is a $\boxed{\text{one-to-one}}$ function on a domain, then f has an inverse function.



$y = x^2$

Not one-to-one



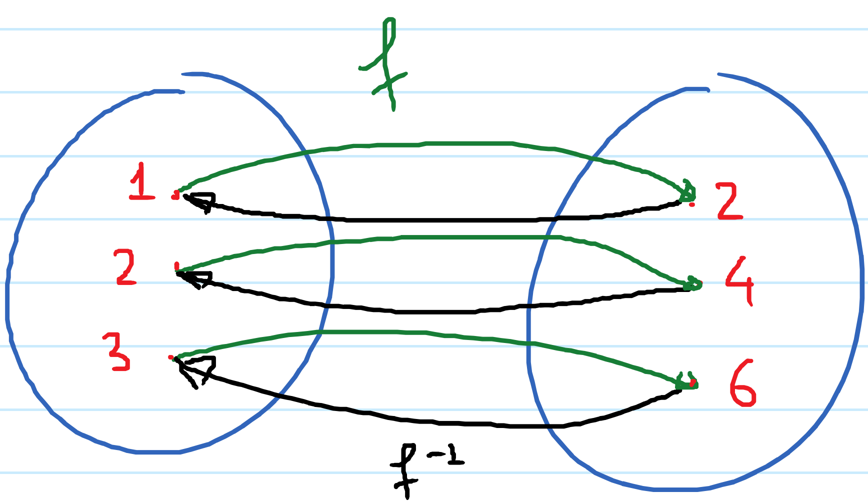
$y = x^3$

One-to-one

The inverse function of f is denoted by $y = f^{-1}(x)$

(read as f inverse of x) (Note: $f^{-1}(x) \neq \dfrac{1}{f(x)}$)

E.g.



$f(1) = 2$

$f^{-1}(2) = 1$

$$(f^{-1})'(5) = \frac{1}{f'(\underbrace{f^{-1}(5)}_{4})}$$

$$= \frac{1}{\boxed{f'(4)}} = \frac{1}{2}$$

2 (Given)

$$(f^{-1})'(5) = \boxed{\frac{1}{2}}$$

Why is the I.F.T true?

Know: $f(f^{-1}(x)) = x$ (this equation holds)

Take the derivative with respect to x of both sides

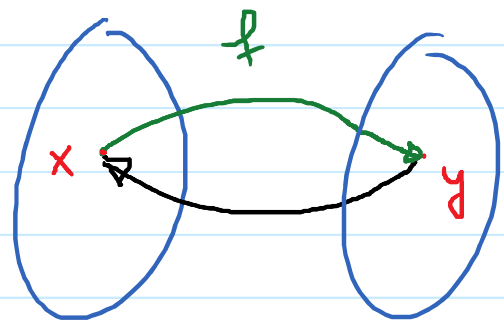
$$\underbrace{f'(f^{-1}(x)) \cdot (f^{-1})'(x)}_{\text{Chain rule}} = 1$$

$$\longrightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \checkmark$$

$$f(x) = y \longleftrightarrow f^{-1}(y) = x$$

So,

$$\boxed{\begin{aligned} f^{-1}(f(x)) &= x \\ f(f^{-1}(x)) &= x \end{aligned}}$$



Inverse Function Theorem.

Let f be a function that is differentiable and one-to-one on an interval containing x. Then

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

E.g. Given: $f(4) = 5$ and $f'(4) = 2$

Q: Find $(f^{-1})'(5)$. (derivative of the inverse function at $x = 5$)

Use the I.F.T to derive the formulas for the derivative of $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$

* Derive the formula for the derivative of arcsin

Start with $f(x) = \sin x$.

$f^{-1}(x) = \arcsin x$

By the I.F.T

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

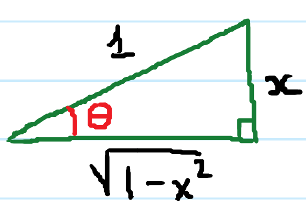
Since $f(x) = \sin x$, $f'(x) = \cos x$

So, $f'(f^{-1}(x)) = \cos(\arcsin x)$

Thus, $$(f^{-1})'(x) = \frac{1}{\cos(\arcsin x)}$$

Goal: simplify this expression; i.e., simplify

$\cos(\boxed{\arcsin x})$. Let $\theta = \arcsin x$. Then $\sin\theta = x$



$\cos(\arcsin x) = \cos\theta = \sqrt{1-x^2}$

Hence, $(f^{-1})'(x) = \dfrac{1}{\sqrt{1-x^2}}$ → this is the formula for the derivative of arcsin function.

In other words,

$$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$$

In Leibnitz notation: $\boxed{\dfrac{d}{dx}[\arcsin x] = \dfrac{1}{\sqrt{1-x^2}}}$

Mixing this with the chain rule: if u is a function of x, then:

$$\boxed{\frac{d}{dx}[\arcsin u] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}}$$

Ex: Derive the formula for the derivative of the arctan function.

Steps: Start with $f(x) = \tan x$.

Use I.F.T. Then simplify.