3.8. Implicit Differentiation

Monday, February 25, 2019   8:05 AM

Goal: Learn to find derivatives implicitly

So far, we are given $y =$ formula in x (explicitly), we are asked to find $\frac{dy}{dx}$ (or y')

E.g. $y = x^3 + 2x - 3 \longrightarrow$ find $\frac{dy}{dx}$?

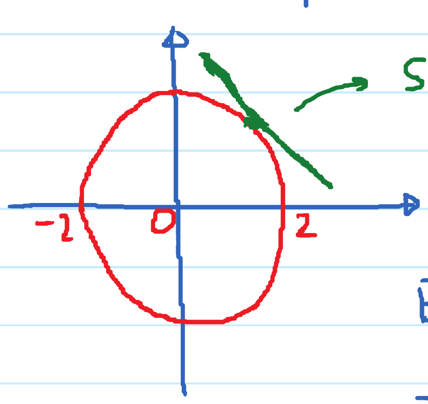
$\longrightarrow$ This is called explicit differentiation.

In many situations, y is not given explicitly in terms of x.

E.g. We could be given an equation that relates y and x but the equation is NOT of the form $y = \ldots$

E.g. $x^2 + y^2 = 4$.

$\longrightarrow$ this is an example where y is given implicitly.



$\longrightarrow$ Slope of tangent line at point $= \frac{dy}{dx}$.

So we need $\frac{dy}{dx}$.

But y is given implicitly.

$\longrightarrow$ Implicit differentiation.

Key: Implicit differentiation is the method of finding $\frac{dy}{dx}$ without having to solve for y by itself first in the equation that relates y and x.

Chain Rule is very important here.

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1} \quad ; \quad \frac{d}{dx}\left[y^n\right] = ny^{n-1} \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\left[\sin(x)\right] = \cos x \quad ; \quad \frac{d}{dx}\left[\sin(y)\right] = \cos(y) \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2} \quad ; \quad \frac{d}{dx}\left[\arctan(y)\right] = \frac{1}{1+y^2} \cdot \frac{dy}{dx}$$

.....

E.g. to illustrate the method of implicit differentiation.

Given the relation: $x^2 + y^2 = 4$.

Q: Use the method of implicit differentiation to find $\frac{dy}{dx}$.

1^{st}: Differentiate both sides w.r.t. x (take $\frac{d}{dx}$ of both sides)

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[4]$$

2nd Apply differentiation rules

$$\underbrace{\frac{d}{dx}[x^2]}_{2x\ \text{(power rule)}}+\underbrace{\frac{d}{dx}[y^2]}_{2y\cdot\frac{dy}{dx}\ \text{(power rule + chain rule)}}=0 \quad \text{(Sum Rule)}$$

$$2x+2y\cdot\frac{dy}{dx}=0$$

power rule power rule + chain rule

$$2x+2y\cdot\boxed{\frac{dy}{dx}}=0$$ → We finished the differentiation process.

3rd Get $\frac{dy}{dx}$ by itself.

$$2\left(x+y\frac{dy}{dx}\right)=0$$

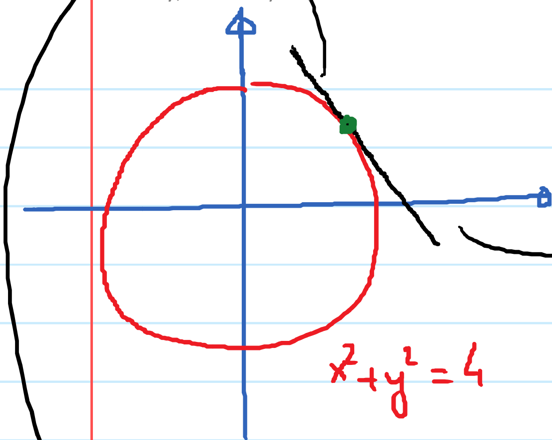
$$x+y\frac{dy}{dx}=0$$

$$y\frac{dy}{dx}=-x$$

$$\boxed{\frac{dy}{dx}=-\frac{x}{y}}$$

Ans: $\boxed{\frac{dy}{dx}=-\frac{x}{y}}$

Q2: Find the equation of the tangent line to graph of $x^2+y^2=4$ at the point $(\sqrt{3},1)$



$x^2+y^2=4$

$$\text{Slope} = \left.\frac{dy}{dx}\right|_{x=\sqrt{3},\, y=1}$$

(I plug $x=\sqrt{3}$, $y=1$ into the formula for $\frac{dy}{dx}$)

$$= -\frac{\sqrt{3}}{1} = \boxed{-\sqrt{3}}$$ Slope

Point-slope formula: $y-1 = -\sqrt{3}\,(x-\sqrt{3})$

$$\longrightarrow \boxed{y = -\sqrt{3}x + 4}$$

equation

E.g. Given: $5x^3 + 3xy = 7x^2 + 5$

Find $\frac{dy}{dx}$.

$$5x^3 + 3xy - 7x^2 = 5$$

$$\frac{d}{dx}\left[5x^3 + 3xy - 7x^2\right] = \frac{d}{dx}[5]$$ (Take $\frac{d}{dx}$ of both sides)

$$\frac{d}{dx}[5x^3] + \frac{d}{dx}[3xy] - \frac{d}{dx}[7x^2] = 0$$ (Sum and difference rule)

$$5\frac{d}{dx}[x^3] + 3\frac{d}{dx}[xy] - 7\frac{d}{dx}[x^2] = 0$$ (Constant multiple rule)

$$15x^2 + 3\overbrace{\left(\underbrace{1}_{\substack{\text{derivative}\\ \text{of } x \text{ w.r.t.}\\ x}}\cdot \underbrace{y}_{\substack{2^{\text{nd}}\\ \text{function}}} + \underbrace{x}_{\substack{1^{\text{st}}\\ \text{function}}}\cdot \underbrace{\frac{dy}{dx}}_{\substack{\text{derivative}\\ \text{w.r.t. } x \text{ of}\\ 2^{\text{nd}} \text{ function}}}\right)}^{\text{product rule}} - 14x = 0$$

$$15x^2 + 3\left(y + x\frac{dy}{dx}\right) - 14x = 0$$

$$15x^2 + 3y + 3x\frac{dy}{dx} - 14x = 0$$

$$3x\frac{dy}{dx} = -15x^2 - 3y + 14x$$

$$\boxed{\frac{dy}{dx} = \frac{-15x^2 - 3y + 14x}{3x}}$$

E.g.

$7xy^3 + 4x^3y = 1$. Find $\frac{dy}{dx}$

$$\frac{d}{dx}\left[7xy^3\right] + \frac{d}{dx}\left[4x^3y\right] = 0 \quad \text{(Sum Rule)}$$

$$7\frac{d}{dx}\left[xy^3\right] + 4\frac{d}{dx}\left[x^3y\right] = 0 \quad \text{(constant multiple rule)}$$

(product + chain)

$$7\left(1\cdot y^3 + x\cdot 3y^2\cdot\frac{dy}{dx}\right) + 4\left(3x^2y + x^3\frac{dy}{dx}\right) = 0$$

$$7y^3 + 21xy^2\frac{dy}{dx} + 12x^2y + 4x^3\frac{dy}{dx} = 0$$

$$21xy^2\boxed{\frac{dy}{dx}} + 4x^3\boxed{\frac{dy}{dx}} = -7y^3 - 12x^2y$$

factor $\frac{dy}{dx}$

$$\frac{dy}{dx}\left(21xy^2 + 4x^3\right) = -7y^3 - 12x^2y$$

$$\boxed{\frac{dy}{dx} = \frac{-7y^3 - 12x^2y}{21xy^2 + 4x^3}}$$

E.g. Find the equation of tangent line to

$$\sin^{-1}(x) + \sin^{-1}(y) = \frac{\pi}{3} \quad \text{at} \quad \left(\boxed{0}, \boxed{\frac{\sqrt{3}}{2}}\right)$$

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) + \frac{d}{dx}\left(\sin^{-1}(y)\right) = 0$$

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0$$

Plug in the point $x=0$, $y=\frac{\sqrt{3}}{2}$ to the equation:

$$1 + \frac{1}{\sqrt{1-\frac{3}{4}}} \cdot \frac{dy}{dx} = 0$$

$= \frac{1}{2}$

$$1 + 2\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{1}{2} \rightarrow \text{Slope.}$$

Equation:

$$\boxed{y = -\frac{1}{2}x + \frac{\sqrt{3}}{2}}$$