4.1. Related Rates

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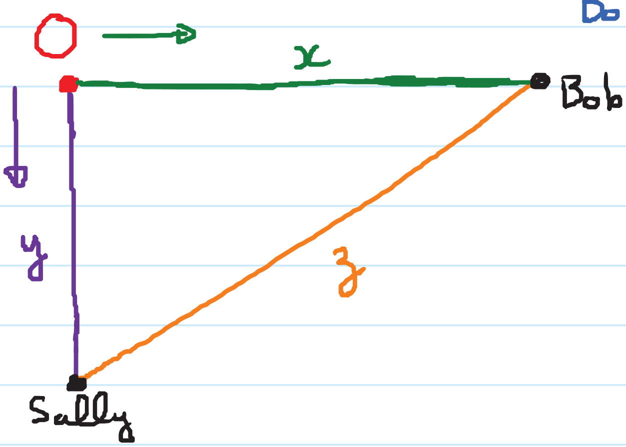
E.g. that illustrates the ideas of related rates problems.

Bob travels East, speed = 4 ft/s

Sally travels South speed is 3 ft/s



x = distance between Bob and O.

y = distance between Sally and O.

z = distance between Bob and Sally

x, y, z are changing with time

So, x, y, z are all functions of time t.

$\left(x = x(t) \;;\; y = y(t) \;;\; z = z(t) \right)$.

derivative

Q: At time $t = 10$ seconds, what is the rate of change of the distance between Bob and Sally?

$\longrightarrow$ Find $\left.\dfrac{dz}{dt}\right|_{t=10}$

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Rate of change of distance

Given: $\boxed{\text{Speed}}$ of Bob $= 4$ ft/s $\rightarrow$ $\boxed{\frac{dx}{dt} = 4 \text{ ft/s}}$

Speed of Sally $= 3$ ft/s $\rightarrow$ $\boxed{\frac{dy}{dt} = 3 \text{ ft/s}}$

Want: $\frac{dz}{dt}$.

Relationship among x, y, z:

$$x^2 + y^2 = z^2 \quad \text{(Pythagorean Theorem)}$$

→ Take the derivative w.r.t. time t of this relation:

Sum, power, chain

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[z^2\right]$$

power rule
chain rule

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$x\underbrace{\boxed{\frac{dx}{dt}}}_{4} + y\underbrace{\boxed{\frac{dy}{dt}}}_{3} = z\underbrace{\boxed{\frac{dz}{dt}}}_{?}$$

At $t = 10$ (s) : $x = 4 \cdot 10 = 40$ ft.

$y = 3 \cdot 10 = 30$ ft.

$$z = \sqrt{x^2 + y^2} = \sqrt{(40)^2 + (30)^2} = 50 \text{ ft}.$$

$$40 \cdot 4 + 30 \cdot 3 = 50 \cdot \left.\frac{dz}{dt}\right|_{t=10}$$

$$\rightarrow \left.\frac{dz}{dt}\right|_{t=10} = \frac{40 \cdot 4 + 30 \cdot 3}{50} = 5 \text{ ft/s.}$$

Rate of change of z at $t=10$

Main components of a related rate problem:

(1) 2 or more quantities involved. The quantities are changing with time, i.e., they are functions of time.

→ Step 1: Identify the quantities involved and identify those that are changing with time. Name them: $x, y, z, u, v, \ldots$

(2) These quantities are related by an equation or a set of equations.

→ Step 2: Find an equation or equation(s) that relate these quantities. (Using given information, draw picture, use geometry...)

(3) We are always given some rates of change in the problem, we are asked to find the missing rates of change.

Step 3: Implicitly differentiate the equation in Step 2 w.r.t. time t. Plug in the rates we know, try to find the missing rates.

E.g. HW #4.

Step 1: Quantities involved: Radius and Area.

Radius = R ; Area = A

Both R and A are changing with time.

Step 2: Relationship between these quatities:

$$A = \pi R^2$$

Step 3: What ROC is given? What ROC is missing?

Given: $\frac{dR}{dt} = 3$ cm/min.

Want: $\frac{dA}{dt} = ?$ when $R = 39$ cm.

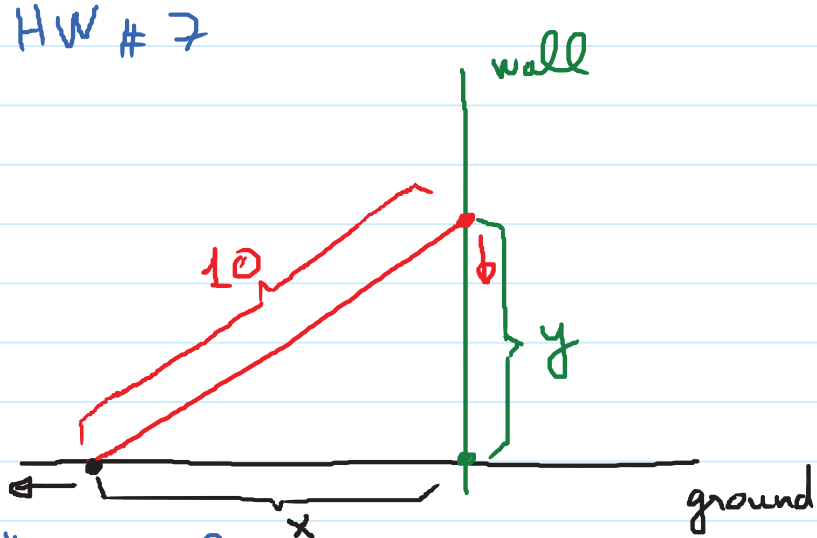
Take $\frac{d}{dt}$ of relation:

$$\frac{d}{dt}[A] = \frac{d}{dt}\left[\pi R^2\right]$$

$$\underbrace{\boxed{\frac{dA}{dt}}}_{?} = \pi \cdot 2\underbrace{\boxed{R}}_{39} \cdot \underbrace{\boxed{\frac{dR}{dt}}}_{3}$$

$$\boxed{\frac{dA}{dt} = 234\pi \ (cm^2/min)}$$

E.g. HW # 7



(1) Quantities involved.

Distance from foot of ladder to wall $= x$

length of ladder $= 10$ (does not change with time)

Distance from top of ladder to ground = y

x, y are changing with time.

(2) Relation:

$$x^2 + y^2 = 100.$$

(3) Given: $\frac{dx}{dt} = 0.4$ m/s

Missing: $\frac{dy}{dt} = ?$ when $x = 6$.

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[100\right]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\underbrace{\boxed{x}}_{6}\ \underbrace{\boxed{\frac{dx}{dt}}}_{0.4} + \underbrace{y}_{?}\ \underbrace{\boxed{\frac{dy}{dt}}}_{?} = 0$$

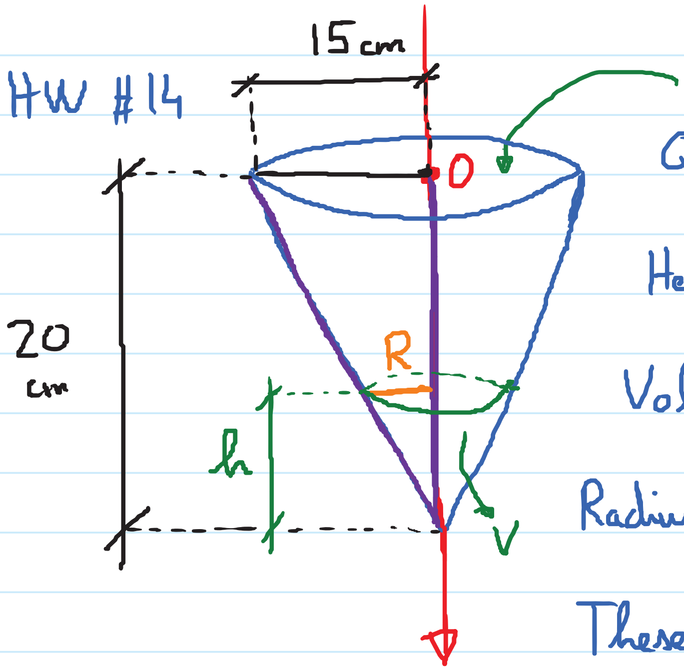
$x^2 + y^2 = 100$; so when $x = 6$: $36 + y^2 = 100$

$y^2 = 64 \rightarrow y = 8.$

$$6\cdot(0.4) + 8\cdot\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-6\cdot(0.4)}{8} = -0.3 \text{ m/s.}$$

HW #14



Quantities involved

Height (level) of water: h

Volume of water : V

Radius (of surface) of water: R

These are changing with time.

Volume of cone $= \frac{1}{3}$ (base area)(height)

$$V = \frac{1}{3}\pi R^2 \cdot h.$$

Given: $\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec}$

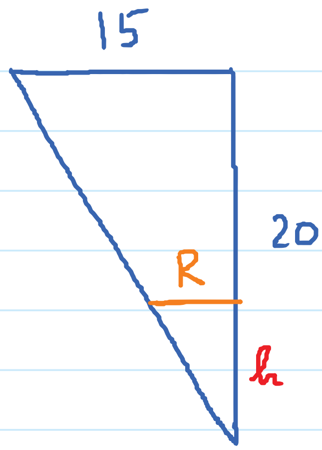
Want: $\frac{dh}{dt}$. when $h = 3$

Take $\frac{d}{dt}$ of relation:

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2R\frac{dR}{dt}\cdot h + R^2\cdot \boxed{\frac{dh}{dt}}\right)$$

? ?

$$10 = \frac{\pi}{3}\left(2\cdot\frac{9}{4}\cdot\frac{3}{4}\boxed{\frac{dh}{dt}} + \frac{81}{16}\boxed{\frac{dh}{dt}}\right)$$



$$\frac{R}{15} = \frac{h}{20}$$

$$R = \frac{15h}{20} = \frac{3h}{4}.$$

$$\rightarrow \frac{dR}{dt} = \frac{3}{4}\frac{dh}{dt}.$$

when $h = 3$; $\frac{R}{15} = \frac{3}{20} \rightarrow R = \frac{45}{20} = \frac{9}{4}$