4.10. Antiderivative

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Definition of an antiderivative of a function.

f: function defined on an interval I.

An antiderivative of f is a function $F(x)$ such that

$$F'(x) = f(x) \text{ for every } x \text{ in } I.$$

In short, an antiderivative of a function is a function whose derivative is equal to the given function.

E.g. Given: $f(x) = x$; defined on $I = (-\infty, \infty)$

Q: Find an antiderivative of f?

A: $F(x) = \frac{x^2}{2}$

Check answer: $\frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{\cancel{2}} \cdot \cancel{2}x = x$

So, $F'(x) = f(x) \rightarrow$ answer is correct

A: $F(x) = \frac{x^2}{2} + 10.$

$F'(x) = x = f(x) \rightarrow$ answer is correct

In general, any antiderivative of $f(x) = x$ will have the form $F(x) = \frac{x^2}{2} + C$; where C is a constant.

E.g. $f(x) = x^2$, defined on $I = (-\infty, \infty)$

Find an antiderivative of f.

A: $F(x) = \frac{x^3}{3}$.

Check: $F'(x) = \left[\frac{x^3}{3}\right]' = \frac{1}{\cancel{3}} \cdot \cancel{3} x^2 = x^2$

In general, any antiderivative of $f(x) = x^2$ have the form $F(x) = \frac{x^3}{3} + C$ where C is a constant.

The formula $\boxed{F(x) = \frac{x^3}{3} + C}$ is called the most general antiderivative of $f(x) = x^2$ or in short, it is called "the" antiderivative of $f(x) = x^2$

E.g. $f(x) = x^{2019}$.

Q: Find the most general antiderivative of $f(x)$?

A: $\boxed{F(x) = \frac{x^{2020}}{2020} + C, \quad C : \text{any constant}}$

→ Q: $f(x) = x^n$.

Find the most general antiderivative of f?

A: $F(x) = \frac{x^{n+1}}{n+1} + C$; C is any constant.

True only if $n \neq -1$

Note: when $n = -1$; $f(x) = x^{-1} = \frac{1}{x}$.

Find the most general antiderivative of $f(x)$?

So, $F(x) = \ln x + C$. ; C: any constant.

Reason: $(\ln x + C)' = \frac{1}{x}$

Very Important notation:

The notation: $\boxed{\int f(x)\,dx = \text{the most general antiderivative of } f(x)}$

Basically,
$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is such that $F'(x) = f(x)$ and C is a constant.

E.g. We have seen:

$$\int x\,dx = \frac{x^2}{2} + C$$

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

(Anti-power rule)

And in general: $\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \; ; \; n \neq -1}$

Also, $\int \frac{1}{x}\,dx = \ln|x| + C$

E.g. Find $\int \sqrt{x}\,dx \underset{\text{Rewrite}}{=} \int x^{\overset{n}{\boxed{\frac{1}{2}}}}\,dx \underset{\text{anti-power rule}}{=} \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \boxed{\frac{2}{3} \cdot x^{\frac{3}{2}} + C}$$

E.g. Find $\int \frac{1}{x^2}\,dx \underset{\text{Rewrite}}{=} \int x^{\overset{n}{\boxed{-2}}}\,dx \underset{\text{antipower rule}}{=} \frac{x^{-2+1}}{-2+1} + C$

$$= \frac{x^{-1}}{-1} + C = \boxed{-\frac{1}{x} + C}$$

Table of useful Antiderivatives

Function $f(x)$	Antiderivative : $\int f(x)\,dx$
$f(x) = x^n$; $n \neq -1$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$; $n \neq -1$
$f(x) = \frac{1}{x}$ ($n=-1$ case)	$\int \frac{1}{x}\,dx = \ln\|x\| + C$
$f(x) = \sin x$	$\int \sin x\,dx = -\cos x + C$
$f(x) = \cos x$	$\int \cos x\,dx = \sin x + C$
$f(x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$f(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$f(x) = \csc^2(x)$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$f(x) = \csc(x)\cot(x)$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

$f(x) = \csc^2(x)$ | $\int \csc^2(x)\,dx = -\cot(x) + C$

$f(x) = \csc(x)\cot(x)$ | $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$