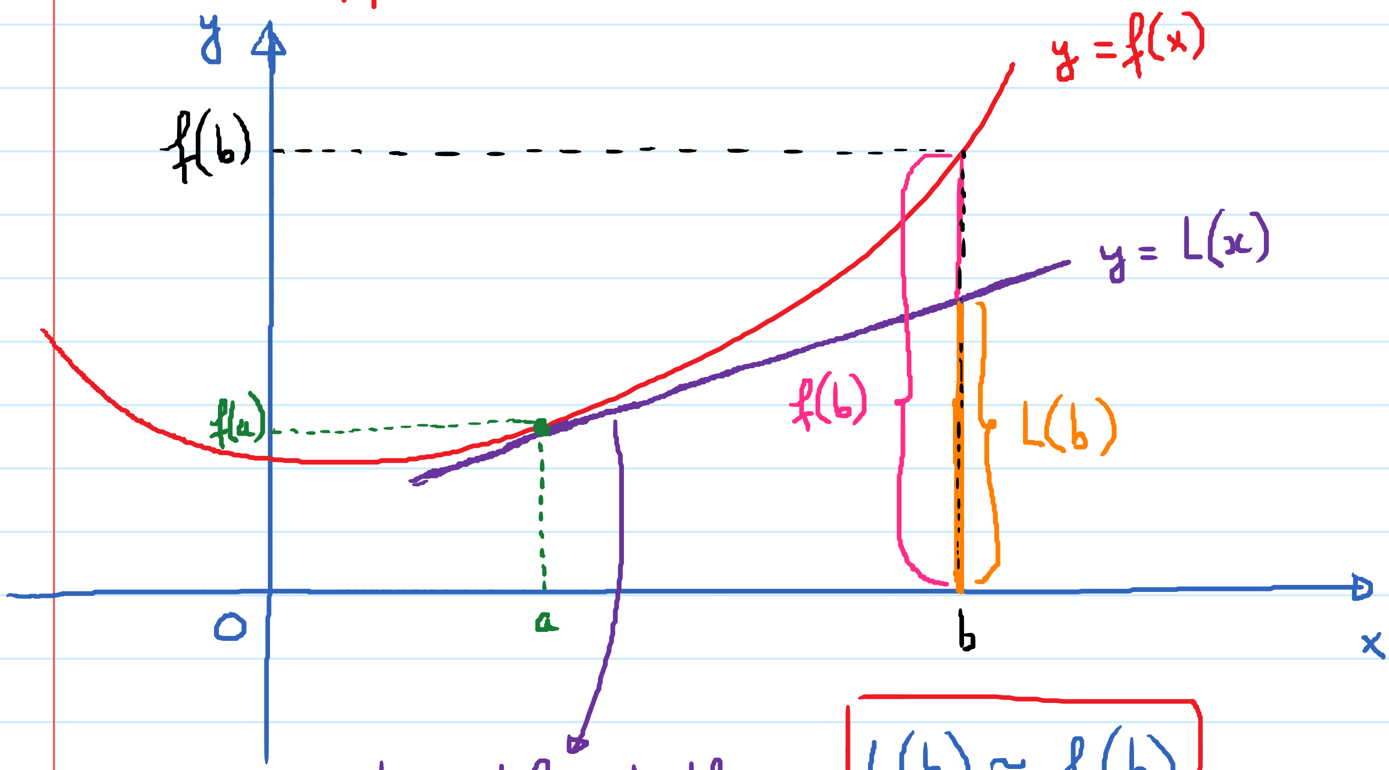
4.2. Linear Approximations and Differentials

Goals: (1) Apply the formula for the linear approximation to a function

(2) Understand the concept of the differential.

(1) Linear Approximation:



tangent line to the graph of $y = f(x)$ at $(a, f(a))$

$\boxed{L(b) \approx f(b)}$

$L(b)$ is the linear approx. to $f(b)$

Goal: Find the formula for $y = L(x)$ (the linear approximation to $y = f(x)$ near $x = a$)

→ find the equation of the tangent line to $y = f(x)$ at $(a, f(a))$.

Slope $= f'(a)$ Point : $(a, f(a))$

Point-slope equation of the tangent line :

$$y - f(a) = f'(a)(x-a)$$

$$\to y = f(a) + f'(a)(x-a)$$

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

This is the formula for the linear approximation $L(x)$ to the function $f(x)$ near the point a.

E.g. Given $f(x) = \sqrt{x}$.

(a) Find the linear approx. to f near the point $a = 9$.

Sol: $L(x) = f(9) + f'(9)(x-9)$

$f(9) = \sqrt{9} = 3$. $f'(x) = \dfrac{1}{2\sqrt{x}} \to f'(9) = \dfrac{1}{6}$

$L(x)$ … 3 … 1 … $(x - 9)$ - Linear approx. to $y = \sqrt{x}$

$$\boxed{L(x) = 3 + \frac{1}{6}(x-9)}$$ → Linear approx. to $y=\sqrt{x}$ near 9.

(b) Use this approximation to estimate the value of $\sqrt{9.1}$

Approximation $= L(9.1) = 3 + \frac{1}{6}(9.1 - 9)$

$= 3 + \frac{1}{6}(0.1) = 3.01667$

So $\sqrt{9.1} \approx 3.01667$

Ex. Use a linear approximation to estimate $\sqrt[3]{1001}$.

(Step 1: Find function $y = f(x)$. Step 2: Find $L(x)$ (decide a)

Step 3: Apply $L(x)$)

$x^{\frac{1}{3}}$

Function: $f(x) = \boxed{\sqrt[3]{x}}$, $a = 1000$.

Linear approximation: $L(x) = f(1000) + f'(1000)(x - 1000)$

$f(1000) = \sqrt[3]{1000} = 10$

$f'(x) = \frac{1}{3} \cdot x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}}$

$f'(1000) = \frac{1}{3(1000)^{2/3}} = \frac{1}{300}$

$\boxed{L(x) = 10 + \frac{1}{300}(x - 1000)}$ → linear approx. to $y = \sqrt[3]{x}$ near 1000

Approximation: $L(1001) = 10 + \frac{1}{300}(1001 - 1000)$

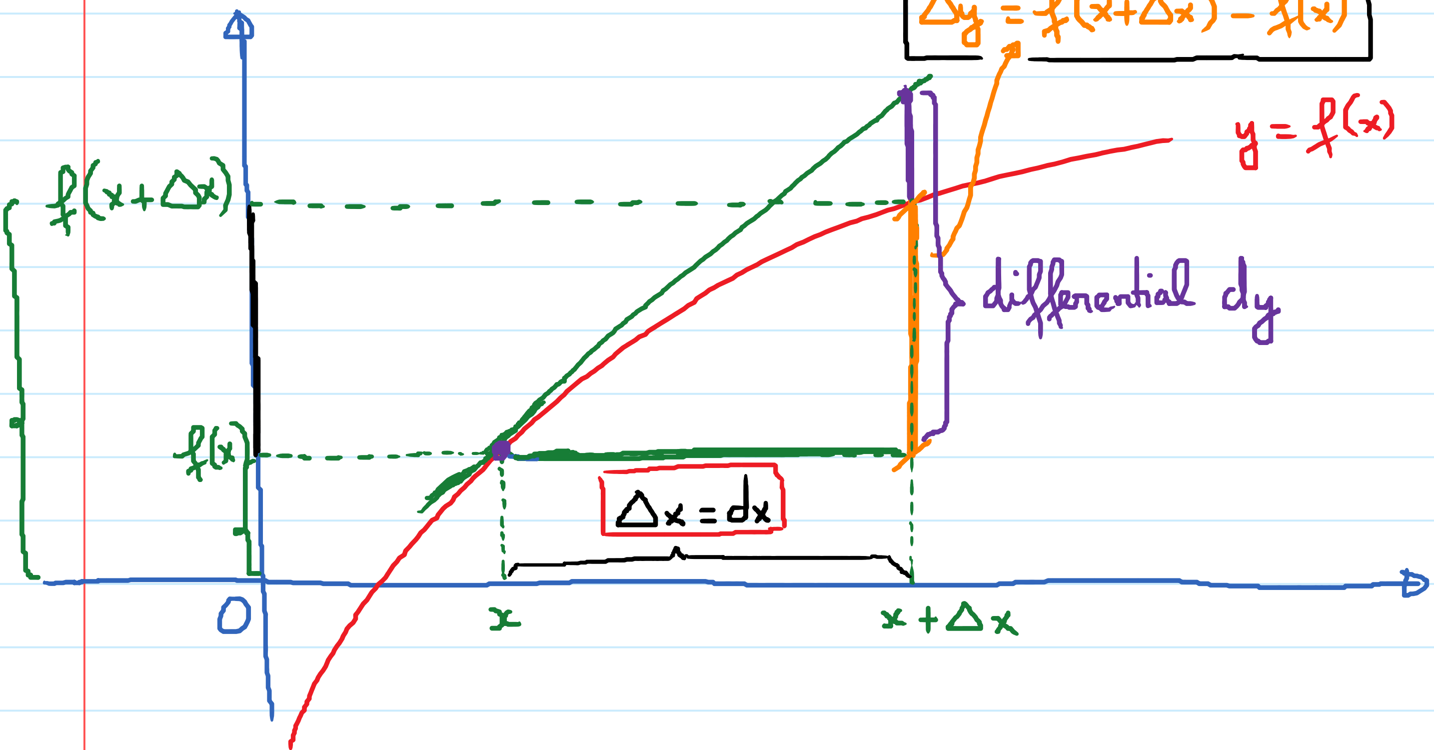
$L(1001) = 10.00333\ldots$

$\sqrt[3]{1001} \approx 10.00333\ldots$

② The differential of a function.

actual change in y

$\boxed{\Delta y = f(x+\Delta x) - f(x)}$



$\frac{dy}{dx}$ = Slope of tangent line at $x = f'(x)$

$\frac{dy}{dx} = f'(x) \longrightarrow \boxed{dy = f'(x) \cdot dx}$

this is the formula for the differential of the function f

at the point x

dy can be used to approximate the actual change Δy of the function as we move from x to $x+\Delta x$.

$$\boxed{dy \approx \Delta y}$$

Summary: $dx = \Delta x$ (change in x)

$\Delta y = f(x+\Delta x) - f(x)$ (Actual change in y)

$dy = f'(x) \cdot dx$ (differential)

$dy \neq \Delta y$. (dy and Δy are different but they are close)

E.g. Given $f(x) = x^3 + x^2 - 2x + 1$.

Find dy and Δy as x changes from 2 to 2.05.

Sol:

$$\Delta y = f(\underbrace{x+\Delta x}_{2.05}) - f(\underbrace{x}_{2}) \qquad (\Delta x = dx = 0.05)$$

$$\Delta y = f(2.05) - f(2)$$

$$= \left((2.05)^3 + (2.05)^2 - 2.05 + 1\right) - \left(2^3 + 2^2 - 2.2 + 1\right)$$

$$\simeq 0.71763$$

$$dy = f'(x) \cdot dx \qquad f'(2) = 14$$

$$x = 2;\ dx = 0.05,\quad f'(x) = 3x^2 + 2x - 2$$

$$\text{So, } dy = f'(2) \cdot (0.05)$$

$$dy = (14) \cdot (0.05) = \boxed{0.7} \rightarrow \text{differential}$$

Note: $\underbrace{dy}_{0.7} \simeq \underbrace{\Delta y}_{0.71763}$.