4.7. Applied Optimization Problems

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An optimization problem is a problem that asks you to find the maximum or minimum of some quantity.

We need to find a function $f(x)$ to describe that quantity.

Case 1: Variable x belongs to a closed interval $[a,b]$

→ Closed interval method

I Find all critical numbers of f within $[a,b]$

II Evaluate f at the critical #(s) and at the endpoints to find max/min value.

Case 2: Variable x belongs to open interval, say $(0,\infty)$ or $(-\infty,\infty)$

→ First derivative → sign chart → find local max/min, end behavior of f to determine abs. max/min.

HW # 1.

Volume of box = L · W · H

$$f(x) = (12-2x)^2 \cdot x$$

→ Find max. on $[0, 6]$ → closed interval method

$$f'(x) = 2(12-2x)\cdot(-2)\cdot x + (12-2x)^2 \cdot 1 = 0$$

$$\rightarrow (12-2x)(-4x + 12 - 2x) = 0$$

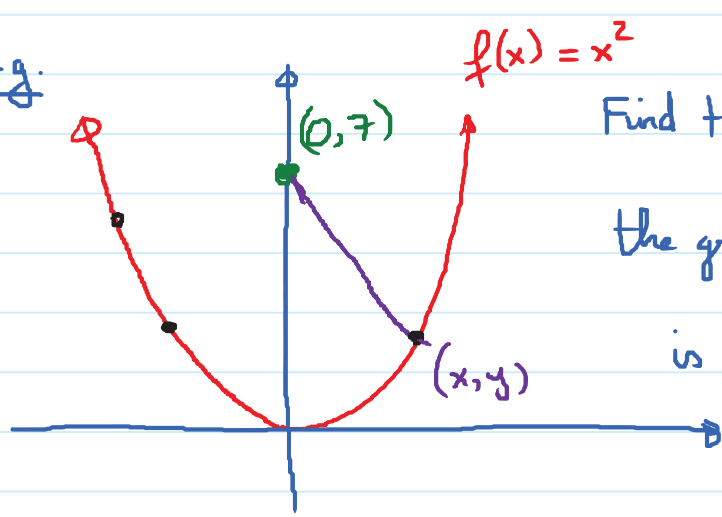
$$\rightarrow (12-2x)(-6x+12) = 0$$

$$\rightarrow x = 6 \,;\, 2 \rightarrow \text{critical \#s.}$$

$$f(0) = 0 \quad ; \quad f(6) = 0$$

$$f(2) = \boxed{128} \rightarrow \text{max. value.}$$

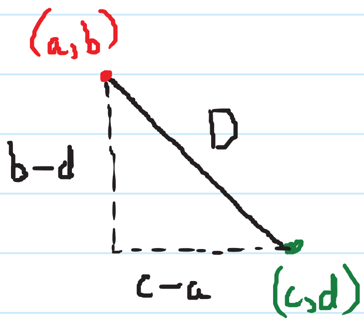
E.g.



Find the point(s) (x,y) on the graph of $f(x) = x^2$ that is closest to $(0,7)$

Distance formula to find distance between 2 points

(a,b) ; (c,d)



$$D = \sqrt{(c-a)^2 + (b-d)^2}$$

For this problem: 2 points: $(0,7)$; (x,y)

$$\boxed{D = \sqrt{x^2 + (y-7)^2}}$$

→ Find Min. of this quantity.

on $(-\infty, \infty)$

Since (x,y) is on parabola, $y = x^2$.

$$\longrightarrow D = \sqrt{\boxed{x^2 + (x^2-7)^2}}$$

x is in $(-\infty, \infty)$

Trick: Rather than finding the min of the quantity with the square root, we can find the min of the quantity under the square root. Then take the square root of result at the end.

→ Find min. of $f(x) = x^2 + (x^2 - 7)^2$; x is in $(-\infty, \infty)$

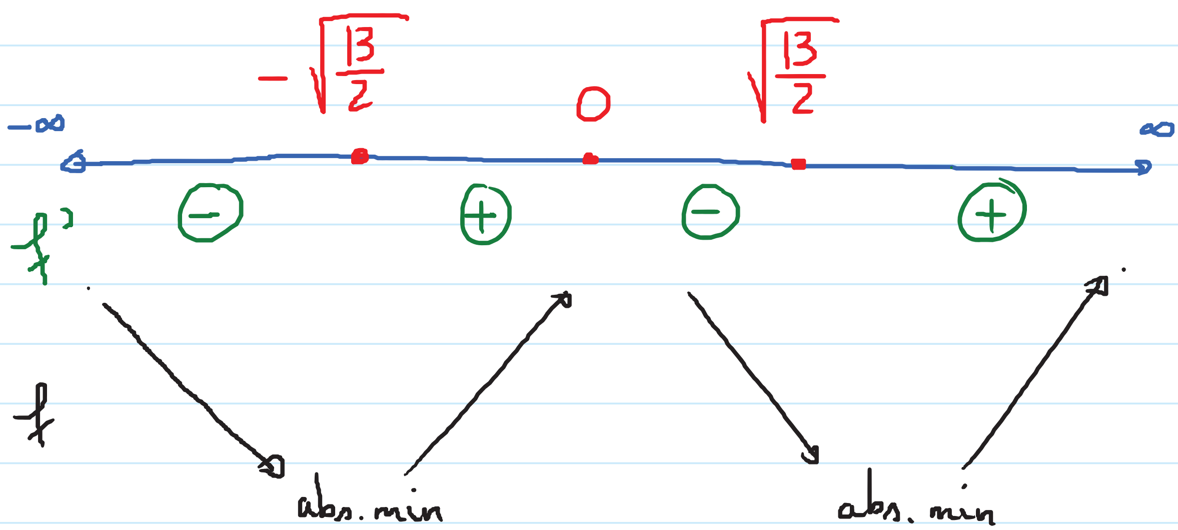
→ First Derivative Test:

$$f'(x) = 2x + 2(x^2 - 7) \cdot 2x = 0$$

$$\rightarrow 2x\left[1 + 2x^2 - 14\right] = 0$$

$$\rightarrow 2x \cdot (2x^2 - 13) = 0$$

$$\rightarrow x = 0 \ ; \quad x^2 = \frac{13}{2} \rightarrow x = \pm\sqrt{\frac{13}{2}}.$$



function has abs. min at $x = -\sqrt{\frac{13}{2}}$; $x = \sqrt{\frac{13}{2}}$.

$$f(x) = x^2 + (x^2 - 7)^2 = \frac{13}{2} + \left(\frac{13}{2} - 7\right)^2$$

$$= \frac{13}{2} + \frac{1}{4} = \frac{27}{4}.$$

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Conclusion: Min. distance $= \dfrac{\sqrt{27}}{2}$.

And it occurs at the points where $x = \pm\sqrt{\dfrac{13}{2}}$

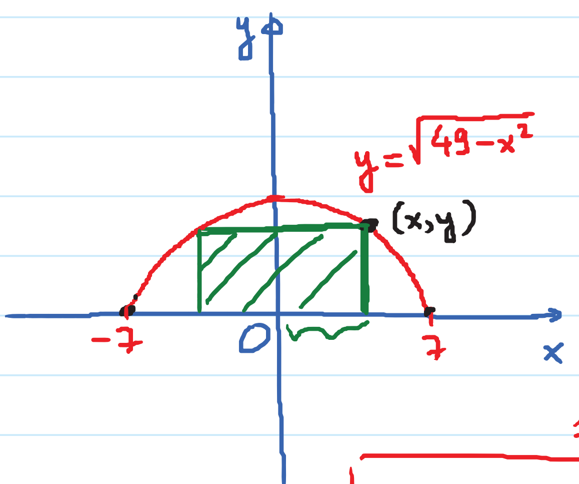
$y = x^2 = \dfrac{13}{2}$.

Points that are closest to $(0,7)$ are $\boxed{\left(\sqrt{\dfrac{13}{2}}, \dfrac{13}{2}\right)}$ or

$\boxed{\left(-\sqrt{\dfrac{13}{2}}, \dfrac{13}{2}\right)}$

E.g.

Max Area = ?



$f(x)$

Area $= 2xy = \boxed{2x\sqrt{49-x^2}}$; x is in $[-7, 7]$

Find max

→ Closed interval method.

$$f'(x) = \frac{2(49-3x^2)}{\sqrt{49-x^2}}$$ II

I $$\frac{2(49-2x^2)}{\sqrt{49-x^2}}$$

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$$\left(\sqrt{u}\right)' = \frac{1}{2\sqrt{u}} \cdot u'$$

$$f(x) = \boxed{2x}\boxed{\sqrt{49-x^2}}$$

$$f'(x) = 2\sqrt{49-x^2} + \cancel{2}x \cdot \frac{(-2x)}{\cancel{2}\sqrt{49-x^2}}.$$

$$= \frac{2\sqrt{49-x^2}\cdot\sqrt{49-x^2}}{1\cdot\sqrt{49-x^2}} - \frac{2x^2}{\sqrt{49-x^2}}$$

$$= \frac{2(49-x^2) - 2x^2}{\sqrt{49-x^2}}$$

$$f'(x) = \frac{2(49-2x^2)}{\sqrt{49-x^2}} = 0$$

$$\longrightarrow 2(49-2x^2) = 0 \qquad \text{critical \#s.}$$

$$\longrightarrow x^2 = \frac{49}{2} \longrightarrow x = \pm\frac{7}{\sqrt{2}}.$$

Also, $f'(x)$ is undefined when denom $= 0$

$$\longrightarrow x = \pm 7.$$

$$f(x) = 2x\sqrt{49-x^2}.$$

$$f(7) = 0, \quad f(-7) = 0 \quad \Big| \quad f\left(-\frac{7}{\sqrt{2}}\right) = -42$$

$$f\left(\frac{7}{\sqrt{2}}\right) = 2\cdot\frac{7}{\sqrt{2}}\cdot\sqrt{49-\frac{49}{2}} = \frac{14}{\sqrt{2}}\cdot\frac{7}{\sqrt{2}} = \frac{84}{2} = 42$$

max.

Max occurs when $x = \frac{7}{\sqrt{2}}$.

$$y = \sqrt{49 - x^2} = \sqrt{49 - \frac{49}{2}} = \frac{7}{\sqrt{2}}$$

$$L = 2 \cdot \frac{7}{\sqrt{2}} = 7\sqrt{2} \qquad W = \frac{7}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$$

HW #9.

$$f(x) = x \cdot \left(\frac{5-x}{3+x}\right) \to \frac{5x - x^2}{3+x} \text{ ; restriction } x \text{ is in } [0,5]$$

$$f'(x) = \frac{(5-2x)\cdot(3+x) - 1\cdot(5x - x^2)}{(3+x)^2}$$

$$= \frac{15 + 5x - 6x - 2x^2 - 5x + x^2}{(3+x)^2}$$

$$= \frac{-x^2 - 6x + 15}{(3+x)^2}.$$

$f'(x) = 0$ when $-x^2 - 6x + 15 = 0$

$\longrightarrow$ quadratic formula $\longrightarrow x = \ldots$

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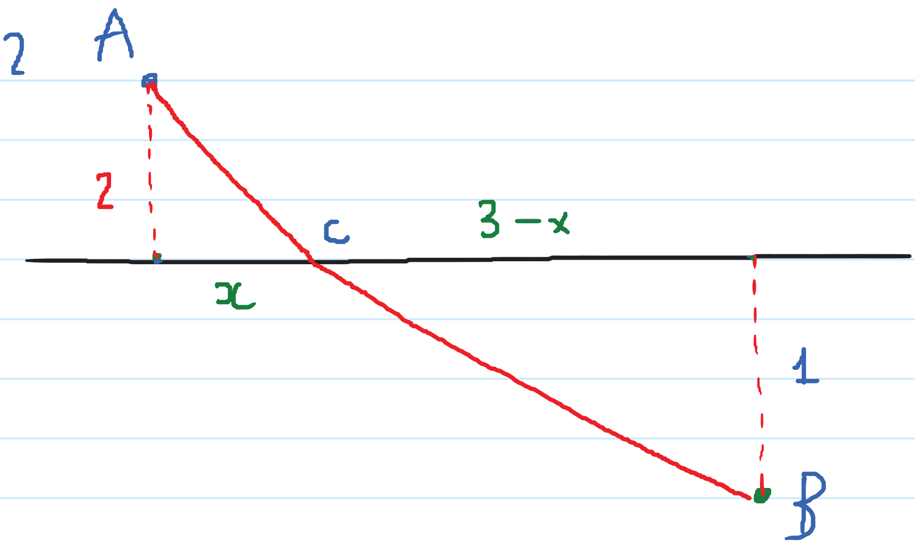

Distance AC $= \sqrt{4+x^2}$

Time from A to C $= \dfrac{\sqrt{4+x^2}}{2}$

Distance CB $= \sqrt{(3-x)^2+1}$

Time from C to B $= \dfrac{\sqrt{(3-x)^2+1}}{4}$

Total time $T = \dfrac{\sqrt{4+x^2}}{2} + \dfrac{\sqrt{(3-x)^2+1}}{4}$

x is in $[0,3]$.

$\longrightarrow$ Minimize.