4.8. L'Hopital Rule

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<u>Obj:</u> 1 Use L'Hopital Rule to find Limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

2 Use L'Hopital rule to find limits of other indeterminate forms.

* Limits of forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$

<u>E.g.</u> $\lim\limits_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 9}$ $\left(\frac{0}{0} \text{ form}\right)$ $\qquad \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}(x+3)} = \frac{4}{6}$

→ L'Hopital way of finding this limit.

$$= \lim_{x \to 3} \frac{2x - 2}{2x} = \frac{4}{6} = \boxed{\frac{2}{3}} \longrightarrow \text{answer.}$$

take derivative
of top, bottom separately

L'Hopital Rule for $\frac{0}{0}$ Limits

Suppose that we are finding a limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ where

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$; i.e., the limit is of the form $\frac{0}{0}$.

L'Hopital rule says that $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

E.g. $\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ ($\frac{0}{0}$ → L'Hopital rule applies)

$$= \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}} - \frac{-1}{2\sqrt{1-x}}}{1} = \lim_{x\to 0} \frac{\frac{1}{2} + \frac{1}{2}}{1} = \boxed{1}$$

Ex ① $\lim_{x\to 1} \frac{x^{15} - 1}{x^{11} - 1}$ $\left(\frac{0}{0}\right)$ $= \lim_{x\to 1} \frac{15x^{14}}{11x^{10}} = \boxed{\frac{15}{11}}$

② $\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$ $\left(\frac{0}{0}\right)$ $= \lim_{x\to 0} \frac{2\cos(2x)}{3\cos(3x)} = \boxed{\frac{2}{3}}$

(3) $\lim_{x \to 0} \dfrac{7x + e^{x} - e^{8x}}{1 - \cos(3x)}$ $\left(\dfrac{0}{0}\right) \rightarrow$ L'Hopital applies

$$= \lim_{x \to 0} \frac{7 + e^{x} - 8e^{8x}}{3\sin(3x)} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{e^{x} - 64e^{8x}}{9\cos(3x)} = \frac{-63}{9}$$

$$= \boxed{-7}.$$

(4) $\lim_{x \to 0} \dfrac{(1+x)^{n} - 1}{x}$ $\left(\dfrac{0}{0}\right) \longrightarrow$ L'Hopital applies.

$$= \lim_{x \to 0} \frac{n(1+x)^{n-1}}{1} = \boxed{n}$$

(5) $\lim_{x \to 0} \dfrac{9^{x} - 7^{x}}{x}$ $\left(\dfrac{0}{0}\right)$

$$(a^{x})' = a^{x} \cdot \ln(a) \quad ; \quad (a^{u})' = a^{u} \cdot \ln(a) \cdot u'$$

$$= \lim_{x \to 0} \frac{9^{x} \cdot \ln(9) - 7^{x} \cdot \ln(7)}{1} = \boxed{\ln(9) - \ln(7)}$$

* Limits of the form $\dfrac{\infty}{\infty}$

E.g. $\lim_{x \to \infty} \dfrac{\boxed{2x^{2} + 5} \rightarrow \infty}{\boxed{5x^{2} + x + 10} \rightarrow \infty}$ Form $\dfrac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{4x}{10x + 1} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{4}{10} = \frac{4}{10} = \boxed{\frac{2}{5}}$$

L'Hopital Rule for $\frac{\infty}{\infty}$ limit:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \text{ where } \lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

E.g. $$\lim_{x\to\infty} \frac{x^3}{5e^{x/3}} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{3x^2}{\frac{5}{3}e^{x/3}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{6x}{\frac{5}{9}e^{x/3}} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{6}{\frac{5}{27}e^{x/3}} = \frac{6}{\infty}$$

$$= \boxed{0}$$

E.g. $$\lim_{x\to\infty} \frac{e^{x^2}}{4-x^3} \left(\frac{\infty}{-\infty}\right) = \lim_{x\to\infty} \frac{2\cancel{x}e^{x^2}}{-3x^{\cancel{2}}}$$

$$= \lim_{x\to\infty} \frac{2e^{x^2}}{-3x} \left(\frac{\infty}{-\infty}\right) = \lim_{x\to\infty} \frac{4xe^{x^2}}{-3} = \frac{\infty}{-3}$$

$$= \boxed{-\infty}$$

$\frac{0}{0}$ and $\frac{\infty}{\infty}$ are called indeterminate form because the answer to a limit problem of such forms can be anything.

There are 5 other indeterminate forms:

$$\boxed{\infty - \infty, \quad 0 \cdot \infty, \quad \infty^0, \quad 1^\infty, \quad 0^0}$$

Note: You cannot apply L'Hopital rule directly to these forms as is.

First step is to use some algebraic manipulation to convert them into either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Second step is to apply L'Hopital to the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.