5.1 and 5.2 Areas and Definite Integrals.

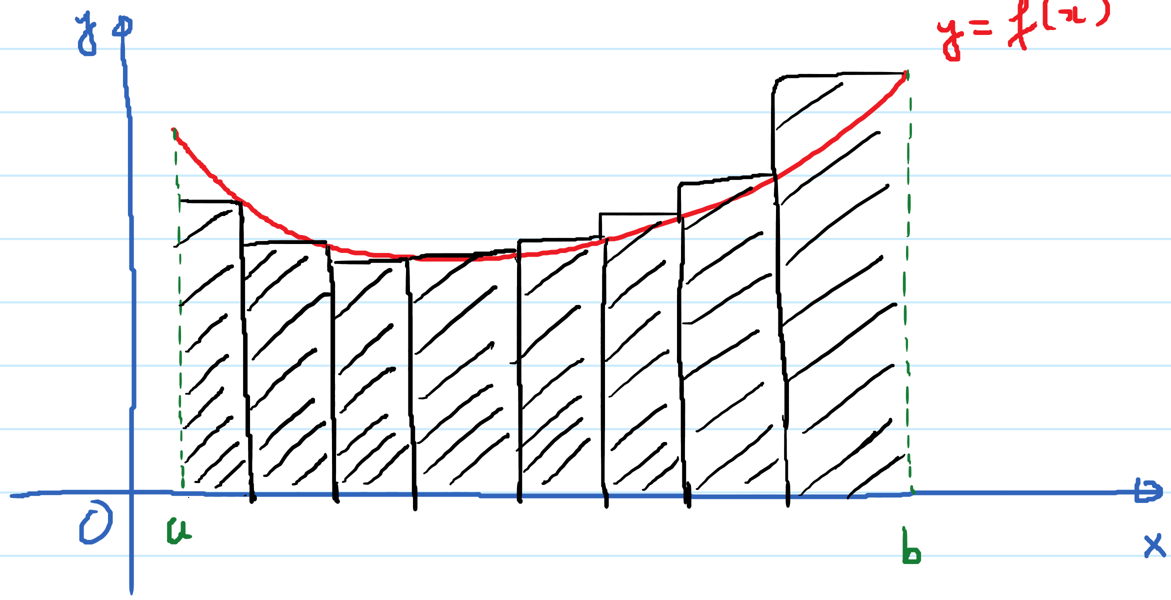
Wednesday, April 17, 2019 8:08 AM

Recall: Antiderivative (Indefinite Integral)

$$\underbrace{\int f(x)\,dx}_{\text{antiderivative of } f(x) \text{ w.r.t. } x} = F(x) + C$$ where $F(x)$ is a function whose derivative is $f(x)$

→ We will develop the concept of the definite integral.

Area Problem:



Find the area under $y = f(x)$; $a \leq x \leq b$

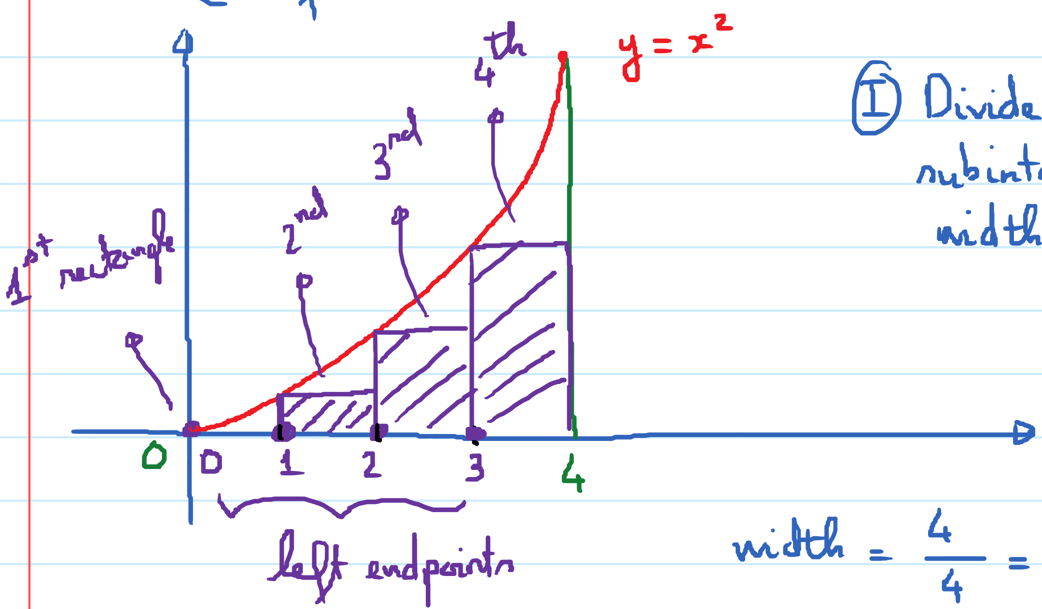
Using the left (Riemann) sum and right (Riemann) sums to approximate areas

Let $f(x) = x^2$; on $[0,4]$

Q: Approximate the area under the graph of $y = x^2$ on $[0,4]$ using L_4 and R_4

L_4 = left endpoints approximation with 4 subintervals

(left Riemann Sum with 4 subintervals)



(I) Divide $[0,4]$ into 4 subintervals of equal width

width $= \frac{4}{4} = 1$

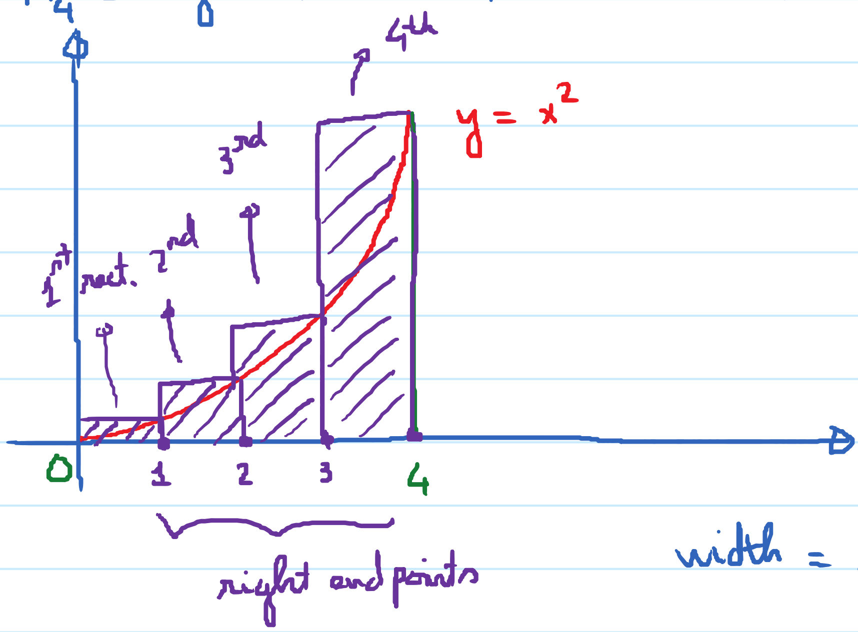
L_4 = Sum of areas of these 4 rectangles:

$= [f(0) + f(1) + f(2) + f(3)] \cdot 1$

$= (0 + 1 + 4 + 9) \cdot 1 = 14$

So, $\boxed{L_4 = 14}$ → underestimate

R_4 = right Riemann sum with 4 subintervals.



width $= \frac{4}{4} = 1$

R_4 = sum of areas of these 4 rectangles:

$= [f(1) + f(2) + f(3) + f(4)] \cdot 1$

$= [1 + 4 + 9 + 16] \cdot 1 = 30$

So, $\boxed{R_4 = 30}$ ⟶ overestimate

A = exact area under the curve.

We know: $14 < A < 30$.

How do we get better approximations for A?

⟶ more rectangles.

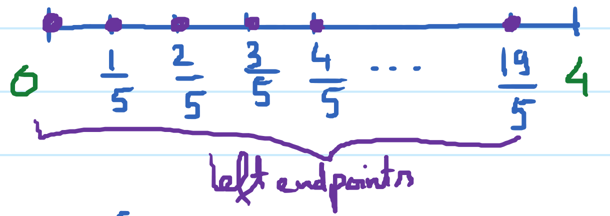
Consider dividing $[0,4]$ into $n = 20$ subintervals.

Find L_{20}, R_{20}?

* L_{20}

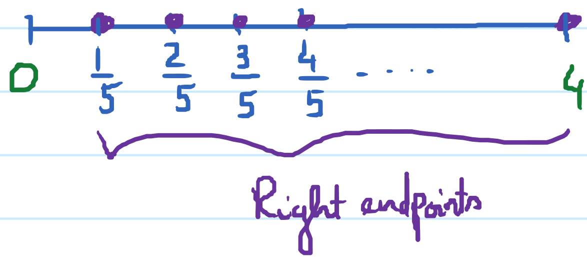
width of a subinterval:

$$\Delta x = \frac{4}{20} = \frac{1}{5}.$$



$$L_{20} = \left[f(0) + f\left(\tfrac{1}{5}\right) + f\left(\tfrac{2}{5}\right) + \dots + f\left(\tfrac{19}{5}\right)\right] \cdot \frac{1}{5} = 19.76$$

* R_{20}



$$R_{20} = \left[f\left(\tfrac{1}{5}\right) + f\left(\tfrac{2}{5}\right) + \dots + f(4)\right] \cdot \frac{1}{5} = 22.96$$

So,

$$19.76 < A < 22.96$$

* What if we use $n = 50$ subintervals?

width $= \Delta x = \dfrac{4}{50} = \dfrac{2}{25}$.

$L_{50} = 20.6976$, $R_{50} = 21.9776$

So, $20.6976 < A < 21.9776$

$\longrightarrow$ first digit of A is 2....

For $n = 100$ subintervals: width $= \Delta x = \frac{4}{100} = \frac{1}{25}$.

$L_{100} = 21.0144$; $R_{100} = 21.6544$

$\rightarrow 21.0144 < A < 21.6544$

$\rightarrow A \simeq 21\ldots$ something

How do we find the exact area A ?

Step 1: Take n to be arbitrary; i.e, we will divide $[0,4]$ into n subintervals where n is considered a variable.

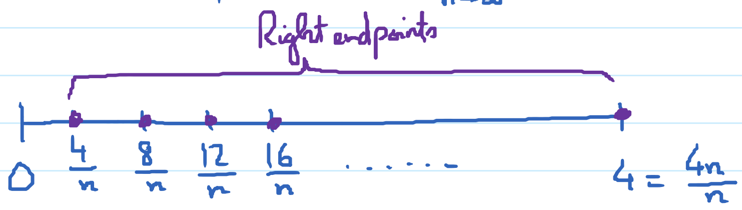
Step 2: Find a formula (in terms of n) for L_n or R_n

Step 3: $A = \lim\limits_{n \to \infty} L_n = \lim\limits_{n \to \infty} R_n$.

(Why? $L_n < A < R_n$ at each n.

As $n \to \infty$, same # by Squeeze Theorem.

* We will find R_n and $\lim_{n\to\infty} R_n$.



Divide $[0,4]$ into n subintervals.

Width of each subinterval $\Delta x = \frac{4}{n}$.

1^{st} right endpoint is $\frac{4}{n}$.
2^{nd} right endpoint is $\frac{8}{n}$.
⋮

In general, the i^{th} right endpoint will be:

$\frac{4i}{n}$.

So,

$\frac{4}{n}\cdot 1 \quad \frac{4}{n}\cdot 2 \quad \cdots \quad \frac{4}{n}\cdot i \quad \cdots \quad \frac{4}{n}\cdot n$

$$R_n = \underbrace{\left[f\left(\frac{4}{n}\right) + f\left(\frac{8}{n}\right) + \cdots + f\left(\frac{4i}{n}\right) + \cdots + f\left(\frac{4n}{n}\right)\right]}_{\text{Sum of heights of all rectangles}} \cdot \underbrace{\frac{4}{n}}_{\text{width of each rect.}}$$

We will use Summation notation to rewrite R_n. $(\sum)$

$$\left(\sum_{i=1}^{n}(3i^2+2i) = (3+2)+(3\cdot4+2\cdot2)+(3\cdot9+2\cdot3)+\cdots+(3n^2+2n)\right)$$