$$R_n = \sum_{i=1}^{n} \boxed{f\left(\frac{4}{n}\cdot i\right)} \cdot \frac{4}{n}.$$

$f(x) = x^2$

So, $f\left(\frac{4}{n}\cdot i\right) = \left(\frac{4}{n}\cdot i\right)^2 = \boxed{\frac{16i^2}{n^2}}$

$$R_n = \sum_{i=1}^{n} \frac{16i^2}{n^2}\cdot\frac{4}{n} = \sum_{i=1}^{n} \boxed{\frac{64i^2}{n^3}} = \frac{64}{n^3}\cdot\boxed{\sum_{i=1}^{n} i^2}$$

There is a formula to calculate $\sum_{i=1}^{n} i^2$.

Formula: $\sum_{i=1}^{n} i^2 = \boxed{\frac{n(n+1)(2n+1)}{6}}$.

So,

$$R_n = \frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

The exact area A is $A = \lim_{n\to\infty} R_n$

degree = 3

$$A = \lim_{n\to\infty} \frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{128}{6} = \frac{64}{3}$$

degree = 3

So, the exact area under $y = x^2$ on $[0,4]$ is $\boxed{\frac{64}{3}}$

Main idea of the Riemann Sums.

Use L_n, R_n to find the area under the graph of $y = f(x)$ on $[a, b]$.

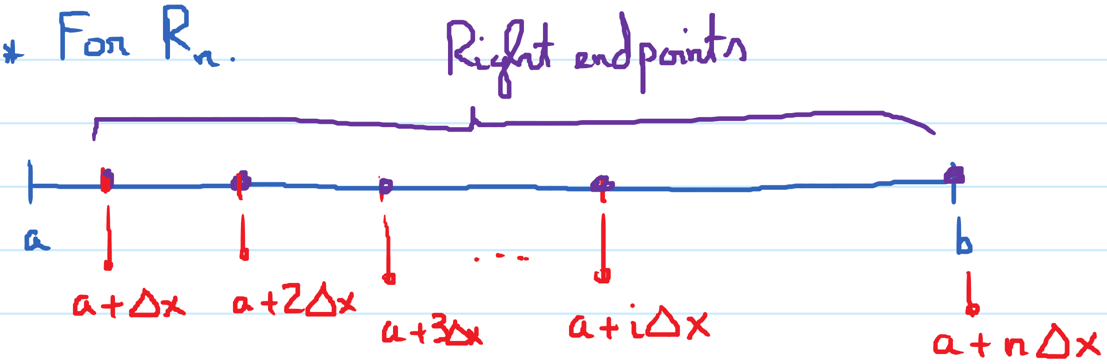
<u>Step 1</u>: Divide $[a, b]$ into n subintervals of equal width.

Width of each subinterval $= \boxed{\Delta x = \dfrac{b-a}{n}.}$

(width of each small rectangle)

<u>Step 2</u>: Find a formula for L_n or for R_n.

* For R_n.

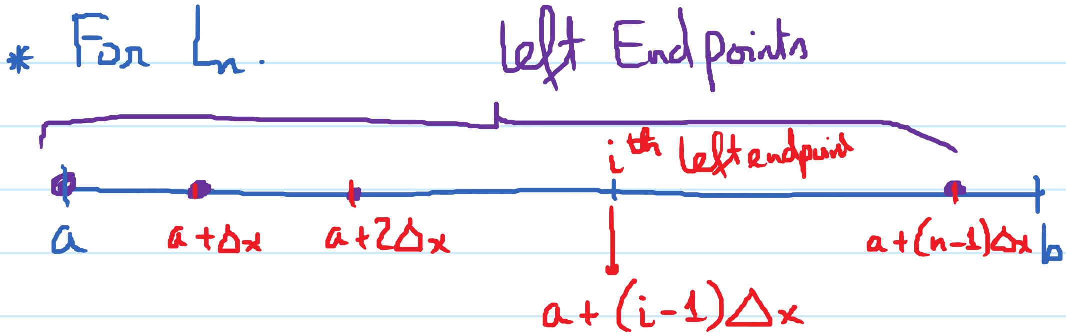


The $i^{\underline{th}}$ right endpoint: $x_i = a + i\Delta x$.

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$\boxed{R_n = \sum_{i=1}^{n} f(a + i\Delta x) \cdot \frac{b-a}{n}.}$$

* For L_n.



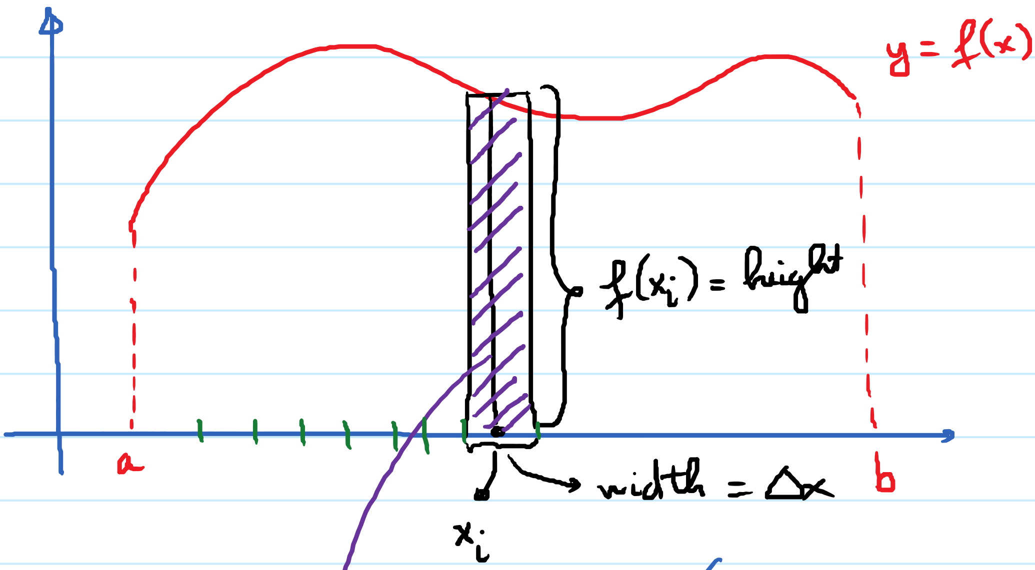
i^{th} left endpoint: $x_i = a + (i-1)\Delta x$

So, $L_n = \sum_{i=1}^{n} f(x_i)\cdot \Delta x$

$$\boxed{L_n = \sum_{i=1}^{n} f\left(a+(i-1)\Delta x\right)\cdot \frac{b-a}{n}.}$$

Step 3: Exact area $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$.

Note:



Area $= f(x_i)\cdot \Delta x$ (x_i: any point within subinterval)

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = A.$$

Very important Notation:

The limit above is called the definite integral of $y = f(x)$ on $[a, b]$.

Notation:

upper bound

$$\int_{\boxed{a}}^{\boxed{b}} \boxed{f(x)}\,\boxed{dx} = \text{exact area under the graph of } y = f(x) \text{ on } [a,b]$$

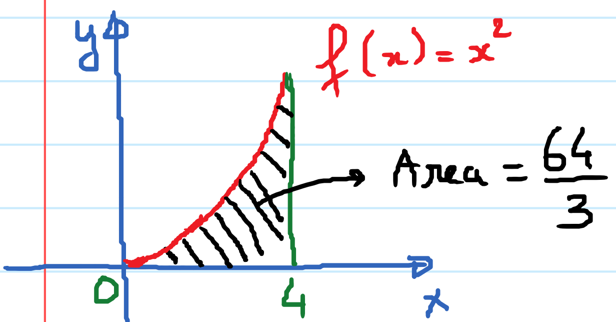
lower bound

integrand

variable of integration

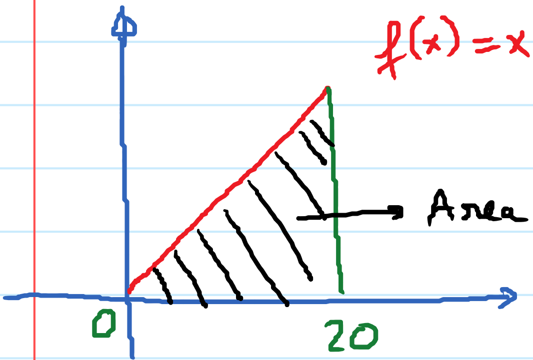
$\int_a^b f(x)\,dx$: read as the definite integral of $f(x)$ from a to b with respect to x.

E.g.



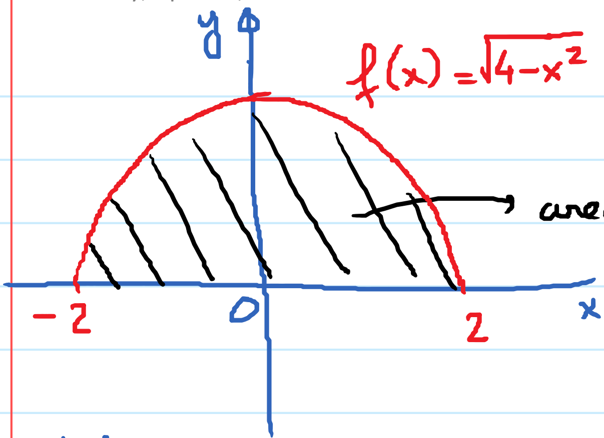
So, $\int_0^4 x^2\,dx = \frac{64}{3}$

Find $\int_0^{20} f(x)\,dx = 200$



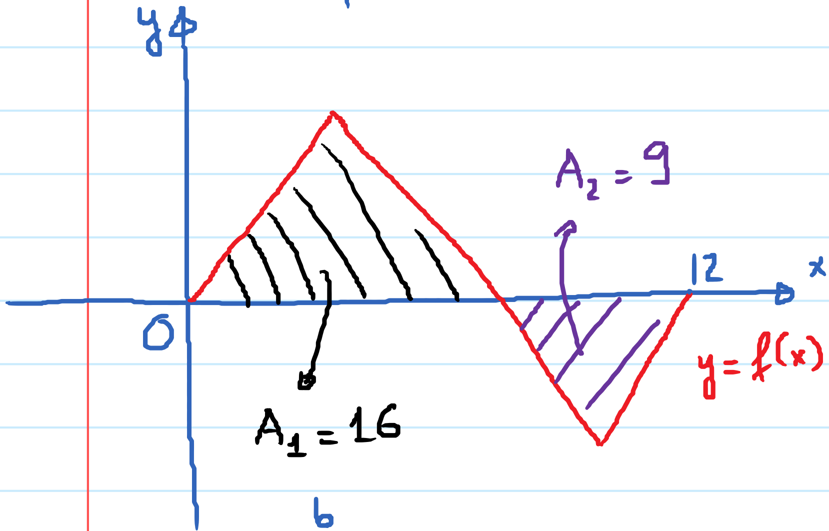
Area $= \frac{\text{base} \cdot \text{height}}{2} = \underset{20}{200}$

So $\int_0^{20} x\,dx = 200$



Find $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{\pi\cdot(2)^2}{2} = 2\pi$

area of semicircle



$\displaystyle\int_{0}^{12} f(x)\,dx = A_1 - A_2$

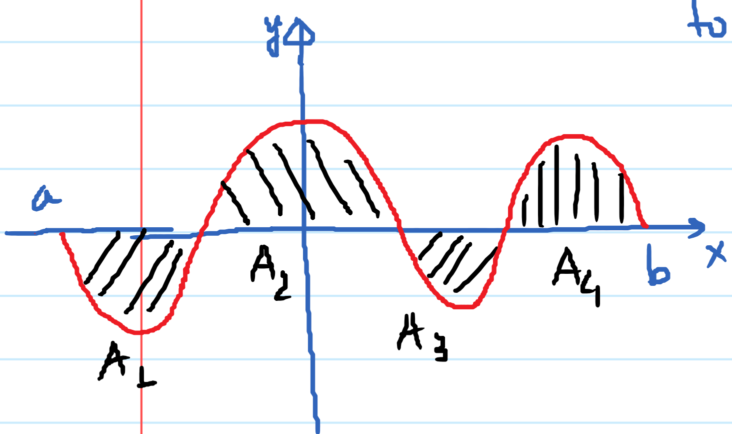
$= 16 - 9 = \boxed{7}$

(We subtract A_2 b/c f is under x-axis)

$\displaystyle\int_{a}^{b} f(x)\,dx =$ "signed" area

if $f < 0$, area < 0.

→ In general; $\displaystyle\int_{a}^{b} f(x)\,dx$ gives us the signed area between $f(x)$ and x-axis from $x=a$ to $x=b$.



$\displaystyle\int_{a}^{b} f(x)\,dx = -A_1 + A_2 - A_3 + A_4$

Wednesday, April 17, 2019 9:33 AM

Useful Properties of Definite Integrals.

① $$\int_a^a f(x)\,dx = 0$$

② $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

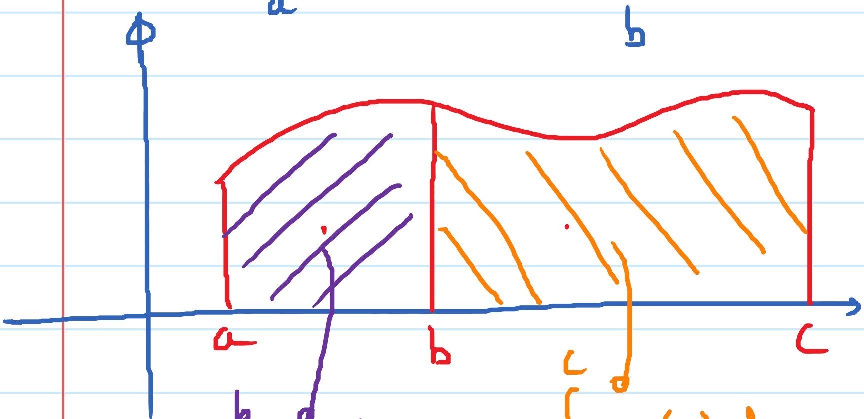
(Switch the bounds → change the sign)

③ $$\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$$

(Can pull a constant out of the integral.

④ $$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

⑤ $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \text{whole area} = \int_a^c f(x)\,dx$$

HW: $f(x) = x^2$; $[1, 2]$

L_{50}. $\Delta x = \frac{b-a}{n} = \frac{1}{50}$

$x_i = a + (i-1)\Delta x = 1 + (i-1)\cdot\frac{1}{50}$

$$L_{50} = \sum_{i=1}^{50} f\left(1 + \frac{i-1}{50}\right)\cdot\frac{1}{50}$$

$$= \sum_{i=1}^{50} \left(1 + \frac{i-1}{50}\right)^2\cdot\frac{1}{50}$$

HW: $f(x) = \sin x$; $[0, \pi]$

R_{30}. $\Delta x = \frac{\pi}{30}$.

$x_i = a + i\Delta x = 0 + i\cdot\frac{\pi}{30} = i\cdot\frac{\pi}{30}$.

$$R_n = \sum_{i=1}^{30} f(x_i)\Delta x = \sum_{i=1}^{30} \sin\left(i\cdot\frac{\pi}{30}\right)\cdot\frac{\pi}{30}.$$