5.3. The Fundamental Theorem of Calculus

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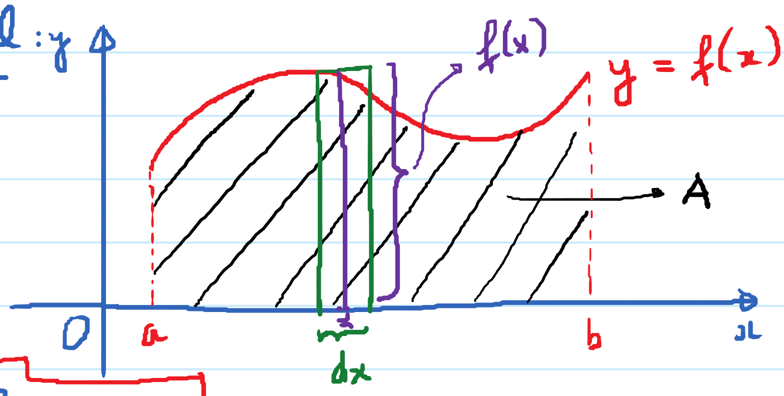
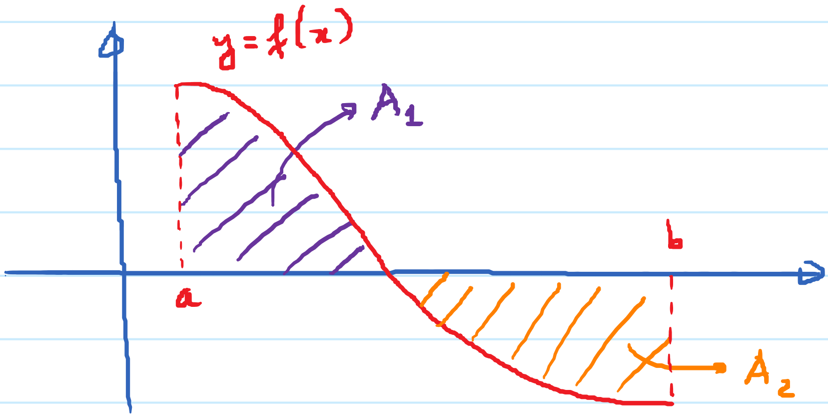
Recall:



$$\boxed{\int_a^b \underbrace{f(x)}_{\text{height}}\,\underbrace{dx}_{\text{width}}}$$ = Area under the curve $y = f(x)$ over the interval $[a, b]$

Definite integral from a to b of $f(x)$ w.r.t. x

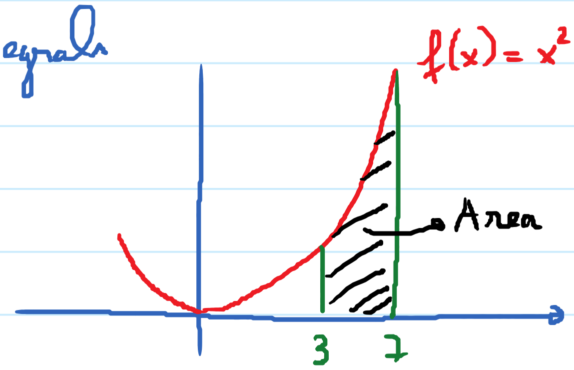


$$\int_a^b f(x)\,dx = \text{signed area between } f(x) \text{ and } x\text{-axis} \; ; \; a \leq x \leq b$$

$$= A_1 - A_2$$

Goal: Use the Fundamental Theorem of Calculus (Part II) to find / evaluate definite integrals

E.g. $\displaystyle\int_3^7 x^2\,dx$



Step 1: Find the antiderivative:

$$\int x^2\,dx = \boxed{\frac{x^3}{3}} + C \longrightarrow F(x)$$

(Here we used the formula $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C;\ n \neq -1$)

Note: Since we are interested in the definite integral and not the general antiderivative, we ignore the constant C and choose a particular antiderivative, call it $F(x)$.

$$F(x) = \frac{x^3}{3}.$$

Step 2: Evaluate $F(x)$ at the upper bound and the lower bound of the definite integral and find the difference.

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$$\underset{\text{upper bound}}{F(\boxed{7})} - \underset{\text{lower bound}}{F(\boxed{3})} = \frac{(7)^3}{3} - \frac{(3)^3}{3}$$

$$= \frac{343}{3} - \frac{27}{3} = \frac{316}{3}$$

So, $\displaystyle\int_3^7 x^2\,dx = \frac{316}{3}$

Shortcut for writing the solution:

$$\int_3^7 x^2\,dx = \underbrace{\left.\frac{x^3}{3}\right|_3^7}_{} = \frac{(7)^3}{3} - \frac{(3)^3}{3} = \frac{316}{3}$$

notation to indicate that you evaluate $\frac{x^3}{3}$ at 7 and at 3 and find the difference

F.T.C. - Part II

If f is a continuous function on $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$; i.e., $F'(x) = f(x)$.

Then:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Notation: (important)

$$F(x)\Big|_a^b := F(b) - F(a)$$

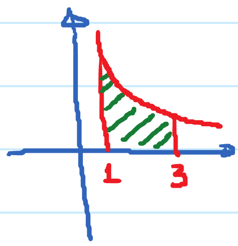
So, FTC - Part II, rewritten:

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b}$$

E.g.

* $$\int_1^3 \frac{1}{x}\,dx = \ln|x|\Big|_1^3 = \ln|3| - \ln|1|$$

$$= \ln 3 - \cancel{\ln 1}^{\,0} = \boxed{\ln 3}$$



* $$\int_6^{12} \sqrt{x}\,dx \overset{\text{Rewrite}}{=} \int_6^{12} x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_6^{12} = \frac{2}{3} \cdot x^{\frac{3}{2}}\Big|_6^{12}$$

$$= \frac{2}{3}\cdot(12)^{\frac{3}{2}} - \frac{2}{3}(6)^{\frac{3}{2}} = \cdots$$

Rewrite

E.g. $$\int_1^4 \left(\frac{4}{x^2} - \frac{1}{x^3}\right)dx = \int_1^4 \left(4x^{-2} - x^{-3}\right)dx$$

$$= \left(4\cdot\frac{x^{-1}}{-1} - \frac{x^{-2}}{-2}\right)\Bigg|_1^4 = \left(-\frac{4}{x} + \frac{1}{2x^2}\right)\Bigg|_1^4$$

$$= \left[-\frac{4}{4} + \frac{1}{2(4)^2}\right] - \left[-\frac{4}{1} + \frac{1}{2(1)^2}\right] = \cdots$$

E.g. $$\int_0^1 (2t-1)^2\,dt = \int_0^1 \left(4t^2 - 4t + 1\right)dt$$

Expand

$$= \left(4\cdot\frac{t^3}{3} - 4\cdot\frac{t^2}{2} + t\right)\Bigg|_0^1 = \left(\frac{4t^3}{3} - 2t^2 + t\right)\Bigg|_0^1$$

$$= \left(\frac{4}{3} - 2 + 1\right) - (0) = \boxed{\frac{1}{3}}$$

E.g. $$\int_1^9 \frac{u-8}{\sqrt{u}}\,du = \int_1^9 (u-8)u^{-1/2}\,du$$

$$= \int_1^9 \left(u^{1/2} - 8u^{-1/2}\right)du = \left(\frac{2}{3}u^{3/2} - 16u^{1/2}\right)\Bigg|_1^9 = -\frac{44}{3}.$$