5.5 - u-substitution method

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E.g. We know: $\int x^{10}\,dx = \dfrac{x^{11}}{11} + C.$

x is just a "dummy" variable.

$$\int u^{10}\,du = \frac{u^{11}}{11} + C.$$

Now, consider: $\int (7x+12)^{10}\,dx$

Make a u-substitution:

Let $\boxed{u = 7x+12}$

Integral becomes $\int u^{10} \boxed{dx}$

Problem!

Need: transform dx into something in terms of u

$$\frac{du}{dx} = 7 \rightarrow du = 7\,dx \rightarrow dx = \frac{du}{7}$$

The original integral now becomes:

$$\int u^{10} \cdot \frac{du}{7} = \frac{1}{7}\int u^{10}\,du = \frac{1}{7}\cdot\frac{u^{11}}{11} + C = \boxed{\frac{(7x+12)^{11}}{77} + C}$$

Key idea:

Step 1: Make a choice for u = function of x.

(base on the form of integrand

Step 2: Find du and solve for dx in terms of du.

Step 3: Transform the original integral to an integral in terms of u ONLY.

Step 4: Apply Basic Antiderivative formulas

(For e.g.

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1) \quad ; \quad \int \frac{1}{u}\,du = \ln|u| + C$$

$$\int \sin(u)\,du = -\cos(u) + C ; \quad \int \cos(u)\,du = \sin(u) + C$$

$$\int e^u\,du = e^u + C \quad ; \quad \int \frac{du}{1+u^2} = \arctan(u) + C \ldots)$$

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E.g. Find $\displaystyle\int \Big(\underbrace{\boxed{7x^2+5}}_{u}\Big)^{2019} \cdot x\, \underbrace{\boxed{dx}}_{\frac{du}{14x}}$

(1) $u = 7x^2 + 5$ (Choose u)

(2) $du = 14x\, dx$

$\longrightarrow dx = \dfrac{du}{14x}$ (Get dx from du)

(3) $\displaystyle\int u^{2019} \cdot \cancel{x} \cdot \frac{du}{14\cancel{x}}$ (Plug in)

$\displaystyle\longrightarrow \frac{1}{14}\int u^{2019}\, du$ (integral that involves only u)

$\longrightarrow \dfrac{1}{14} \cdot \dfrac{u^{2020}}{2020} + C$

$\longrightarrow \boxed{\dfrac{1}{14} \cdot \dfrac{(7x^2+5)^{2020}}{2020} + C.}$

2nd way of doing this:

$u = 7x^2 + 5 \rightarrow du = 14x\, dx$

$\displaystyle\frac{1}{14}\int \Big(\underbrace{7x^2+5}_{u}\Big)^{2019} \underbrace{14x\, dx}_{du} \longrightarrow \frac{1}{14}\int u^{2019}\, du \rightarrow$

E.g. $\int x^2 \cdot \sin(x^3)\,dx = \frac{1}{3}\int \sin(x^3)\,3x^2\,dx$ du

$u = x^3 \longrightarrow du = 3x^2\,dx$

$\frac{1}{3}\int \sin(u)\,du = \frac{1}{3}\left(-\cos(u)\right) + C$

$= -\frac{1}{3}\cos(u) + C = -\frac{1}{3}\cos(x^3) + C.$

E.g. $\int \frac{7x}{(8-x^2)^3}\,dx$ $u = 8 - x^2$; $du = -2x\,dx$

Answer: $\frac{7}{4(8-x^2)^2} + C.$

E.g. $\int \frac{x}{\sqrt{9-x^2}}\,dx$ $u = 9 - x^2$; $du = -2x\,dx$

Answer: $-\sqrt{9-x^2} + C.$

More examples of u-sub.

E.g. $\int \sec^7(2x)\tan(2x)\,dx$.

Rewrite

$u = \sec(2x)$; $du = \sec(2x)\tan(2x)\cdot 2\,dx$

$$\frac{1}{2}\int \underbrace{\boxed{\sec^6(2x)}}_{u^6}\ \underbrace{\boxed{\sec(2x)\tan(2x)\cdot 2\,dx}}_{du}$$

The integral becomes:

$$\frac{1}{2}\int u^6\,du = \frac{1}{2}\cdot\frac{u^7}{7} + C = \boxed{\frac{\sec^7(2x)}{14} + C}$$

E.g. $\int \cos^3(\theta)\,d\theta$

Rewrite

Let $u = \sin(\theta)$. $du = \cos(\theta)\,d\theta$

$$\int \boxed{\cos^2\theta}\cdot\underbrace{\boxed{\cos\theta\,d\theta}}_{du}$$

$\cos^2\theta + \sin^2\theta = 1 \rightarrow \cos^2\theta = 1 - \sin^2\theta$
$= \boxed{1 - u^2}$

$$\int (1-u^2)\,du = u - \frac{u^3}{3} + C$$

$$= \boxed{\sin(\theta) - \frac{\sin^3(\theta)}{3} + C}$$

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E.g. $-1\int x\,(1-x)^{99}\,(-1\,dx)$ $\quad u^{99}$ $\quad du$

$x = 1-u$
(b/c $u = 1-x$ → isolate x)

Let $u = 1-x$. $\quad du = -1\,dx$

$$= -\int (1-u)\,u^{99}\,du$$

$$= -\int \left(u^{99} - u^{100}\right) du = -\frac{u^{100}}{100} + \frac{u^{101}}{101} + C$$

$$= \boxed{-\frac{(1-x)^{100}}{100} + \frac{(1-x)^{101}}{101} + C}$$

* u-substitution with definite integral.

$$\int_0^{\pi/3} \frac{\sin(x)}{\cos^3(x)}\,dx$$

1st method: Ignore the bounds, do the indefinite integral, get answer in terms of x, then plug in bounds into antiderivative as in F.T.C.

2nd method: Update the bounds right after we made the u-sub.

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$$-\int_0^{\pi/3} \frac{\overbrace{-\sin(x)\,dx}^{du}}{\underbrace{\cos^3(x)}_{u^3}}$$

Let $\boxed{u = \cos(x)}$. $du = -\sin(x)\,dx$

Update bounds: Lower bound for x: $\boxed{x = 0}$

→ Lower bound for u: $u = \cos(0) = 1$

Upper bound for x: $\boxed{x = \frac{\pi}{3}}$

→ Upper bound for u: $u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Integral becomes

$$\boxed{-\int_1^{1/2} \frac{du}{u^3}} = \int_{1/2}^{1} u^{-3}\,du = \left.\frac{u^{-2}}{-2}\right|_{1/2}^{1}$$

$$= \left.\left(-\frac{1}{2u^2}\right)\right|_{1/2}^{1}$$

$$= -\frac{1}{2} - \left(-\frac{1}{2\cdot\left(\frac{1}{4}\right)}\right)$$

$$= -\frac{1}{2} + \frac{1}{\frac{1}{2}} = -\frac{1}{2} + 2 = \boxed{\frac{3}{2}}$$