Trig Substitution.

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E.g. 1: Substitution: $x = a\sin\theta$

$$\int \frac{4}{x^2\sqrt{16-x^2}}\,dx$$

let $\boxed{x = 4\cdot\sin\theta} \longrightarrow dx = 4\cos\theta\, d\theta$

$$= \int \frac{4}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}\,4\cos\theta\, d\theta$$

$$= \int \frac{\cancel{16}\cos\theta}{\cancel{16}\sin^2\theta\cdot\sqrt{16(\boxed{1-\sin^2\theta})}}\,d\theta$$

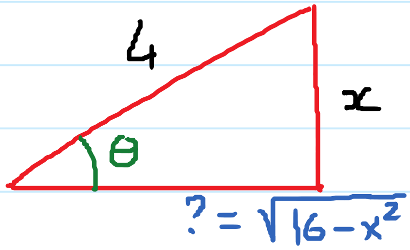
$\cos^2\theta$

$$= \int \frac{\cancel{\cos\theta}}{\sin^2\theta\cdot 4\cdot\cancel{\cos\theta}}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot(\theta) + C$$

$x = 4 \cdot \sin\theta \rightarrow \sin\theta = \frac{x}{4}$ SOH CAH TOA



Want $\cot\theta \longrightarrow \cot\theta = \frac{\text{adj}}{\text{opp}}$

$x^2 + ?^2 = 16$

$\cot\theta = \frac{\sqrt{16-x^2}}{x}$

$? = \sqrt{16-x^2}$

$$\boxed{\int \frac{4}{x^2\sqrt{16-x^2}}\,dx = -\frac{1}{4}\cdot\frac{\sqrt{16-x^2}}{x} + C}$$

Eg. 2

$$\int \frac{x^3}{4\sqrt{4+x^2}}\,\boxed{dx}$$

$\boxed{x = 2\tan\theta} \longrightarrow dx = 2\cdot\sec^2\theta\, d\theta$

$$4\int \frac{\cancel{8}\tan^3\theta}{\cancel{4}\sqrt{4+4\tan^2\theta}}\,\cancel{2}\sec^2\theta\, d\theta$$

$$= 4\int \frac{\tan^3\theta\,\sec^2\theta}{\sqrt{4(\boxed{1+\tan^2\theta})}}\,d\theta \longrightarrow \sec^2\theta$$

$$= 2\int \frac{\tan^3\theta \, \sec^{\cancel{2}}\theta}{\cancel{\sec\theta}}\, d\theta$$

$$= 2\int \tan^3\theta \, \sec\theta \, d\theta$$

$$= 2\int \tan^2\theta \cdot \overset{du}{\boxed{\tan\theta \, \sec\theta \, d\theta}}$$

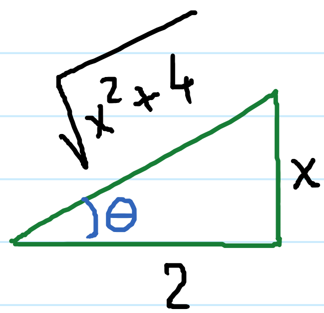
let $u = \sec\theta$

$du = \sec\theta \, \tan\theta \, d\theta$

$$= 2\int (u^2 - 1)\, du$$

$$= 2 \cdot \left(\frac{u^3}{3} - u\right) + C$$

$$= 2\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C .$$



Recall:

$x = 2\tan\theta \rightarrow \tan\theta = \frac{x}{2}$

$$\boxed{\sec\theta = \frac{\sqrt{x^2+4}}{2}}$$

Ans: $2 \cdot \left(\left(\frac{\sqrt{x^2+4}}{2}\right)^3 \cdot \frac{1}{3} - \frac{\sqrt{x^2+4}}{2}\right) + C$

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$$\int \frac{x^3}{4\sqrt{4+x^2}}\,dx = \boxed{2\left(\frac{(x^2+4)^{\frac{3}{2}}}{24} - \frac{\sqrt{x^2+4}}{2}\right) + C}$$

E.g. 3: $\displaystyle\int_4^8 \frac{\sqrt{x^2-16}}{x^2}\,dx$

Let $\boxed{x = 4\sec\theta} \longrightarrow dx = 4\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{16\sec^2\theta - 16}}{\cancel{16}\sec^{\cancel{2}}\theta}\,\cancel{4}\,\cancel{\sec\theta}\tan\theta\,d\theta$$

$$= \frac{1}{4}\int \frac{\sqrt{16(\underbrace{\boxed{\sec^2\theta - 1}}_{\tan^2\theta})}}{\sec\theta}\tan\theta\,d\theta$$

$$= \int \frac{\tan\theta}{\sec\theta}\cdot\tan\theta\,d\theta = \int_0^{\pi/3} \frac{\tan^2\theta}{\sec\theta}\,d\theta.$$

* Find bounds for θ.

$x = 4 \longrightarrow 4 = 4\sec\theta \longrightarrow \sec\theta = 1 \longrightarrow \cos\theta = 1 \rightarrow \boxed{\theta = 0}$

$x = 8 \longrightarrow 8 = 4\sec\theta \longrightarrow \sec\theta = 2 \longrightarrow \cos\theta = \frac{1}{2} \rightarrow \boxed{\theta = \frac{\pi}{3}}$

$$\int_0^{\pi/3} \frac{\tan^2\theta}{\sec\theta}\, d\theta = \int_0^{\pi/3} \frac{\sec^2\theta - 1}{\sec\theta}\, d\theta$$

$$= \int_0^{\pi/3} \left(\frac{\sec^2\theta}{\sec\theta} - \frac{1}{\sec\theta} \right) d\theta$$

$$= \int_0^{\pi/3} (\sec\theta - \cos\theta)\, d\theta$$

$$= \left(\ln\left|\sec\theta + \tan\theta\right| - \sin\theta \right) \Big|_0^{\frac{\pi}{3}}$$

$$= \left(\ln\left|2 + \sqrt{3}\right| - \frac{\sqrt{3}}{2} \right) - \left(\cancel{\ln 1}^{\,0} - 0 \right)$$

$$= \boxed{\ln\left|2 + \sqrt{3}\right| - \frac{\sqrt{3}}{2}}$$

E.g. 4. $\displaystyle\int \frac{x}{\sqrt{x^2-6x+5}}\,dx$

$x^2-6x+5 = x^2-6x+9 \; -9+5$

$= (x-3)^2 - 4$

$\displaystyle\int \frac{x}{\sqrt{(x-3)^2-4}}\,dx$

Let $\boxed{x-3 = 2\sec\theta} \longrightarrow dx = 2\sec\theta\tan\theta\,d\theta$

$\displaystyle\int \frac{2\sec\theta+3}{\sqrt{4\sec^2\theta-4}}\,2\sec\theta\tan\theta\,d\theta$

$\displaystyle= \int \frac{2\sec\theta+3}{\sqrt{4(\boxed{\sec^2\theta-1})}}\,2\sec\theta\tan\theta\,d\theta$

$\tan^2\theta$

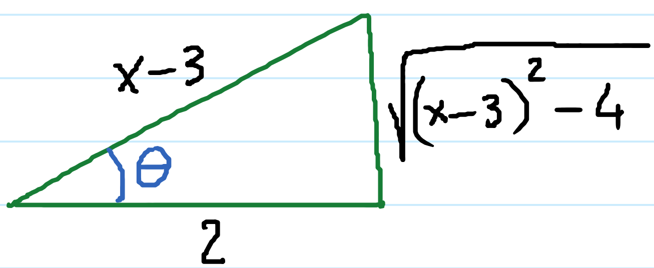
$$= \int \frac{2\sec\theta + 3}{\cancel{2}} \cdot \cancel{2}\sec\theta \, d\theta$$

$$= \int (2\sec\theta + 3)\sec\theta \, d\theta$$

$$= \int (2\sec^2\theta + 3\sec\theta) \, d\theta$$

$$= 2\tan\theta + 3\ln\left|\sec\theta + \tan\theta\right| + C$$

Recall: $x - 3 = 2\sec\theta \longrightarrow \sec\theta = \dfrac{x-3}{2}$



$$\tan\theta = \frac{\sqrt{(x-3)^2 - 4}}{2}$$

$$= \cancel{2} \cdot \frac{\sqrt{(x-3)^2 - 4}}{\cancel{2}} + 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{(x-3)^2 + 4}}{2}\right| + C$$

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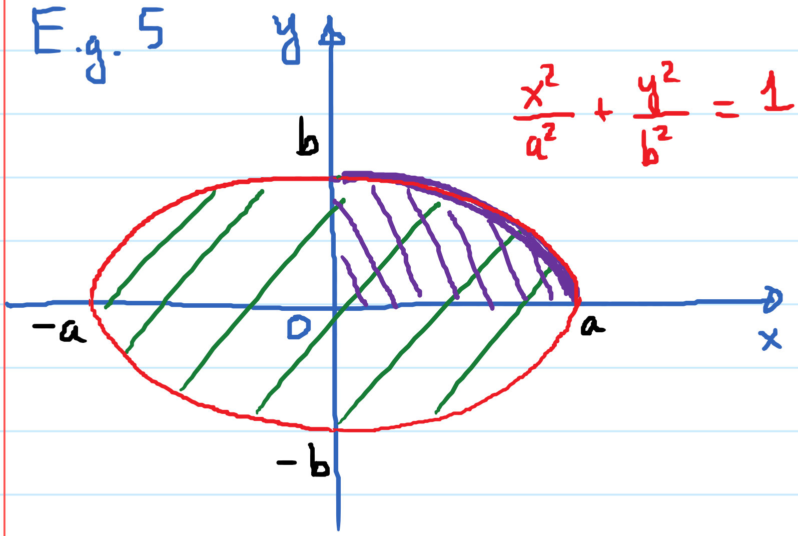
E.g. 5



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\text{Area}=4\int_0^a f(x)\,dx$$

$$=4\cdot\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$$=4b\int_0^a\sqrt{\frac{a^2-x^2}{a^2}}\,dx$$

$$=\frac{4b}{a}\cdot\int_0^a\sqrt{a^2-x^2}\,\boxed{dx}$$

Let $\boxed{x=a\sin\theta}\longrightarrow dx=a\cos\theta\,d\theta$

$$= \frac{4b}{a} \int \sqrt{a^2 - a^2 \sin^2\theta}\; a\cos\theta \, d\theta$$

$$= \frac{4b}{\cancel{a}} \int \sqrt{a^2 (\boxed{1 - \sin^2\theta})}\; a\cos\theta \, d\theta$$

$\cos^2\theta$

$$= 4ba \int_0^{\pi/2} \cos^2\theta \, d\theta$$

Bounds for θ

$x = 0 : \quad 0 = a\sin\theta \longrightarrow \sin\theta = 0 \longrightarrow \theta = 0$

$x = a : \quad a = a\sin\theta \longrightarrow \sin\theta = 1 \longrightarrow \theta = \frac{\pi}{2}$

$$= 4ba \boxed{\int_0^{\pi/2} \cos^2\theta \, d\theta}$$

Wallis' formula $\longrightarrow \frac{\pi}{4}$

$$= 4ba \cdot \frac{\pi}{4} = \boxed{\pi ab}$$