Integration by partial fractions

Integral of the form $\int \frac{P(x)}{Q(x)}\,dx$

(Integrating Rational Functions)

E.g. 2. $\int \frac{5}{x^2+3x-4}\,dx$

Step 1: Factor denominator and decompose

$$\int \boxed{\frac{5}{(x+4)(x-1)}}\,dx$$

→ Case 1: Deno = product of distinct linear factors.

→ Partial Fractions Decomposition.

$$\boxed{\frac{5}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}.}$$

where A and B are to be determined

Step 2: Find A, B.

$$\frac{5}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

Multiply both sides by the common denom. $(x+4)(x-1)$:

$$5 = A(x-1) + B(x+4)$$

<u>Strategy 1</u>: Substitution.

* Substitute $x=1$ to both sides:

$$5 = 5B \longrightarrow \boxed{B=1}$$

* Substitute $x=-4$ to both sides:

$$5 = -5A \longrightarrow \boxed{A=-1}$$

<u>Strategy 2</u>: Undetermined coefficient.

$$5 = A(x-1) + B(x+4)$$

$$5 = Ax - A + Bx + 4B$$

$$5 = (A+B)x + (-A+4B)$$

Coeff. of x on LHS $= 0$. Coeff. of x on RHS $= A+B$

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So, $A + B = 0$

Constant term on LHS $= 5$

Constant term on RHS $= -A + 4B$

So, $-A + 4B = 5$.

So, we have: $\begin{cases} A + B = 0 \\ -A + 4B = 5 \end{cases}$

$A = -1$; $B = 1$ (after we solved)

$$\longrightarrow \frac{5}{(x+4)(x-1)} = \frac{-1}{x+4} + \frac{1}{x-1}$$

Step 3: Integrate:

$$\int \frac{5}{(x+4)(x-1)}\,dx = \int \left(\frac{-1}{x+4} + \frac{1}{x-1} \right) dx$$

$$= -\int \frac{dx}{x+4} + \int \frac{dx}{x-1}$$

$$= \boxed{-\ln|x+4| + \ln|x-1| + C}$$

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Case 2: Repeated linear Factors.

For e.g., after we factor denom. we get

$$(x-1)(x+4)^5$$

$$\frac{P(x)}{(x-1)(x+4)^5} = \frac{A}{x-1} + \frac{B}{x+4} + \frac{C}{(x+4)^2} + \frac{D}{(x+4)^3}$$

$$+ \frac{E}{(x+4)^4} + \frac{F}{(x+4)^5}$$

$\longrightarrow$ determine A, B, C, D, E, F.

E.g. 3: $$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx$$

Step 1: Factor Denom. completely

$$\int \frac{x^2+3x-4}{x(x^2-4x+4)}\,dx = \int \frac{x^2+3x-4}{x(x-2)^2}\,dx$$

Step 2: Decompose:

$$\frac{x^2+3x-4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Multiply both sides by $x(x-2)^2$:

$$x^2+3x-4 = A(x-2)^2 + Bx(x-2) + Cx$$

$$x^2+3x-4 = A(x^2-4x+4) + B(x^2-2x) + Cx$$

$$x^2+3x-4 = (A+B)x^2 + (-4A-2B+C)x + 4A.$$

Equate coefficients of both sides:

$$\begin{cases} A+B = 1 & (\text{coeff. of } x^2) \\ -4A-2B+C = 3 & (\text{coeff. of } x) \\ 4A = -4 & \end{cases}$$

$A=-1$; $B=2$; $C=3$

$$\frac{x^2+3x-4}{x(x-2)^2} = \frac{-1}{x} + \frac{2}{x-2} + \frac{3}{(x-2)^2}$$

Step 3: Integrate:

$$\int \frac{x^2+3x-4}{x(x-2)^2}\,dx = -\int \frac{dx}{x} + 2\int \frac{dx}{x-2} + 3\int \frac{dx}{(x-2)^2}$$

$$= \boxed{-\ln|x| + 2\ln|x-2| - 3(x-2)^{-1} + C}$$

Case 3: Non repeated irreducible quadratic factor.

How do we determine whether the expression ax^2+bx+c is irreducible (not-factorable)?

Ans: $b^2-4ac < 0$

For e.g.

$$\frac{P(x)}{(\underbrace{x^2+2x+5}_{\text{irreducible}})(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+5}$$

E.g. 1

(1) $$\frac{7}{2x^3+3x^2-2x} = \frac{7}{x(2x^2+3x-2)}$$

$$= \frac{7}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

(distinct linear factors)

(2) $$\frac{2x^2+1}{(x-3)^3(5x^2-2x^3)} = \frac{2x^2+1}{(x-3)^3x^2(5-2x)}$$

$$= \frac{A_1}{x-3} + \frac{A_2}{(x-3)^2} + \frac{A_3}{(x-3)^3} + \frac{B_1}{x} + \frac{B_2}{x^2} + \frac{C}{5-2x}$$

(Repeated linear factors)

(3) $$\frac{x^2-1}{x^3+x^2+2x} = \frac{x^2-1}{x(x^2+x+2)}$$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+x+2}$$ (non repeated irreducible quadratic factor)