4. $\dfrac{x^4}{(x^2-x+1)^2(x^4+4x^2+4)}$ (Repeated irr quadratic factor

$$= \frac{x^4}{(\underbrace{x^2-x+1}_{\text{irr.}})^2(\underbrace{x^2+2}_{\text{irr.}})^2}$$

$$= \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{(x^2-x+1)^2} + \frac{Ex+F}{x^2+2} + \frac{Gx+H}{(x^2+2)^2}$$

E.g. 4.

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \frac{2x^2-x+4}{x(\underbrace{x^2+4}_{\text{irr. quadratic}})}\,dx$$

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$\rightarrow$ $2x^2-x+4 = A(x^2+4)+(Bx+C)x$

$\rightarrow$ $2x^2-x+4 = Ax^2+4A+Bx^2+Cx$

$$2x^2 - x + 4 = (A+B)x^2 + Cx + 4A$$

$2 = A + B$

$-1 = C$

$4 = 4A \rightarrow$

$$\boxed{\begin{aligned} B &= 1 \\ C &= -1 \\ A &= 1 \end{aligned}}$$

$u = x^2 + 4$

$du = 2x\,dx$

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx = \int \frac{1}{x}\,dx + \int \frac{x-1}{x^2+4}\,dx$$

$$= \ln|x| + \int \frac{\boxed{x\,dx}^{\frac{du}{2}}}{\underbrace{x^2+4}_{u}} - \int \frac{dx}{x^2+4}$$

$$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

E.g.5. $\displaystyle\int \frac{x^2+6x+4}{x^4+8x^2+16}\,dx$

$$= \int \frac{x^2+6x+4}{(x^2+4)^2}\,dx$$

$$\frac{x^2+6x+4}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$\rightarrow x^2+6x+4 = (Ax+B)(x^2+4) + Cx+D$

$\rightarrow x^2+6x+4 = Ax^3+Bx^2+4Ax+4B+Cx+D$

$\rightarrow x^2+6x+4 = Ax^3+Bx^2+(4A+C)x+4B+D.$

$A=0$; $B=1$

$4A+C=6 \rightarrow C=6$

$4B+D=4 \rightarrow D=0$

$\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$

$u=x^2+4$

$du=2x\,dx$

$$\int \frac{x^2+6x+4}{(x^2+4)^2}\,dx = \boxed{\int \frac{1}{x^2+4}\,dx} + \boxed{\int \frac{6x\,dx}{(x^2+4)^2}}$$

$6x\,dx \rightarrow 3\,du$

Second integral: $\displaystyle\int \frac{3\,du}{u^2} = 3\int u^{-2}\,du$

$$= -3u^{-1}$$

<u>Answer:</u> $\boxed{\dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right) - 3\left(x^2+4\right)^{-1} + C}$

<u>E.g.</u> $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx$

$$= \int \frac{x+4}{x^2+2x+1+4}\,dx = \int \frac{x+4}{(x+1)^2+4}\,dx.$$

$$= \int \frac{x+1+3}{(x+1)^2+4}\,dx = \int \frac{x+1}{(x+1)^2+4}\,dx + 3\int \frac{dx}{(x+1)^2+4}$$

1st integral:

$$\int \frac{\boxed{x+1\ \ dx}}{(x+1)^2+4} \qquad \frac{du}{2}$$

Let $u = (x+1)^2+4$

$du = 2(x+1)\,dx$

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln\left|(x+1)^2+4\right|$$

2^{nd} integral: $3\int \frac{dx}{(x+1)^2+4}$ let $u = x+1$

$du = dx$

$$3\int \frac{du}{u^2+4} = \frac{3}{2}\arctan\left(\frac{u}{2}\right)$$

$$= \frac{3}{2}\arctan\left(\frac{x+1}{2}\right)$$

Ans: $\boxed{\frac{1}{2}\ln\left|(x+1)^2+4\right| + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C}$

E.g. 6.

1 $\int \frac{\boxed{\sec^2(x)\,dx}}{\tan(x)(\tan(x)+1)}$ du

let $u = \tan x$

$du = \sec^2 x\, dx$

$$\int \frac{du}{u(u+1)} = \int \left(\frac{A}{u} + \frac{B}{u+1}\right) du$$

$$\boxed{A = 1;\ B = -1}$$

$$\int\left(\frac{1}{u}-\frac{1}{u+1}\right)du$$

$$= \ln|u| - \ln|u+1| + C$$

$$= \ln|\tan x| - \ln|\tan x + 1| + C$$

2. $$\int \frac{e^x}{(e^{2x}+1)(e^x-1)}\,dx$$ Let $u = e^x$. Then $du = e^x\,dx$

$$\int \frac{du}{(u^2+1)(u-1)} = \int \frac{Au+B}{u^2+1} + \frac{C}{u-1}$$

$$\frac{1}{(u^2+1)(u-1)} = \frac{Au+B}{u^2+1} + \frac{C}{u-1}$$

$$1 = (Au+B)(u-1) + C(u^2+1)$$

$$1 = Au^2 + Bu - Au - B + Cu^2 + C$$

$$1 = (A+C)u^2 + (B-A)u + (-B+C)$$

$A + C = 0$; $B - A = 0$; $-B + C = 1$

Thursday, June 13, 2019 2:34 PM

$$A = -\frac{1}{2}\ ; \quad B = -\frac{1}{2}\ , \quad C = \frac{1}{2}$$

$$\int \frac{1}{(u^2+1)(u-1)}\,du = \int \frac{-\frac{1}{2}u - \frac{1}{2}}{u^2+1}\,du + \int \frac{\frac{1}{2}}{u-1}\,du$$

$$= -\frac{1}{2}\boxed{\int \frac{u+1}{u^2+1}\,du} + \frac{1}{2}\int \frac{du}{u-1}$$

$$\frac{1}{2}\ln|u-1|$$

$$\underbrace{\int \frac{u}{u^2+1}\,du}_{\frac{1}{2}\ln(u^2+1)} + \underbrace{\int \frac{1}{u^2+1}\,du}_{\arctan(u)}$$

Ans: $\boxed{-\frac{1}{4}\ln\left|e^{2x}+1\right| - \frac{1}{2}\arctan(e^x) + \frac{1}{2}\ln\left|e^x-1\right| + C}$