Integration by partial fractions

Key formulas

The method of partial fractions can be used to find integral of rational functions, i.e., integral of the form $\int \frac{P(x)}{Q(x)} dx$.

1. Case 1: The denominator $Q(x)$ can be factored into a product of distinct linear factors

   $$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k).$$

   The form of the partial fraction decomposition is

   $$\frac{P(x)}{Q(x)} = \frac{C_1}{a_1 x + b_1} + \frac{C_2}{a_2 x + b_2} + \cdots + \frac{C_k}{a_k x + b_k},$$

   when we integrate $\frac{P(x)}{Q(x)}$, we can apply the formula $\int \frac{C}{ax+b} = \frac{C}{a} \ln|ax+b|$ for each term of right hand side.

2. Case 2: $Q(x)$ has a repeated linear factor, i.e., it contains a factor of the form $(ax+b)^r$. In this case, the partial fraction decomposition must include

   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}$$

   to account for the repeated factor. To integrate $\frac{C}{(ax+b)^k}$ for $k \geq 2$, we rewrite it as $C(ax+b)^{-k}$ and obtain $\int C(ax+b)^{-k} dx = \frac{C}{a(-k+1)}(ax+b)^{-k+1}$.

3. Case 3: $Q(x)$ has an irreducible quadratic factor $ax^2 + bx + c$ (irreducible means it cannot be factored into product of two distinct linear factors with real coefficients). In this case, the partial fraction decomposition must include a term of the form

   $$\frac{Cx + D}{ax^2 + bx + c}.$$

   Integrating such a term often involves separating the numerator, completing the square, u-sub and integrals that involve inverse trig function.

4. Case 4: $Q(x)$ has a repeated irreducible quadratic factor, i.e., it contains a factor of the form $(ax^2 + bx + c)^r$. In this case, the partial fraction decomposition must include

   $$\frac{C_1 x + D_1}{ax^2 + bx + c} + \frac{C_2 x + D_2}{(ax^2 + bx + c)^2} + \cdots + \frac{C_r x + D_r}{(ax^2 + bx + c)^r}.$$

Note:If the degree of the numerator $P(x)$ is greater than or equal to the degree of the denominator $Q(x)$, we need to do long division before applying the method.

Example 1: Determine the correct form of the partial fraction decomposition

Write the form of the partial fraction decompositon of the given expression. Do not solve for the constants.

1. $\dfrac{7}{2x^3+3x^2-2x}$
2. $\dfrac{2x^2+1}{(x-3)^3(5x^2-2x^3)}$
3. $\dfrac{x^2-1}{x^3+x^2+2x}$
4. $\dfrac{x^4}{(x^2-x+1)^2(x^4+4x^2+4)}$

Solution

Write the solution here

Example 2: Distinct linear factors

Find the integral $\displaystyle\int \frac{5}{x^2+3x-4}$.

Solution

Write the solution here

Example 3: Repeated linear factors

Find the integral $\int \frac{x^2+3x-4}{x^3-4x^2+4x}$.

Solution

Write the solution here

Example 4: Irreducible quadratic factors

Find the integral $\int \frac{2x^2-x+4}{x^3+4x}$.

Solution

Write the solution here

Example 5: Repeated irreducible quadratic factors

Find the integral $\int \frac{x^2+6x+4}{x^4+8x^2+16}$.

Solution

Write the solution here

Example 6: u-sub and partial fractions

Find the given integrals

1. $\int \frac{\sec^2(x)}{\tan(x)(\tan(x)+1)}dx$
2. $\int \frac{e^x}{(e^{2x}+1)(e^x-1)}dx$.

Solution

Write the solution here