Lecture 11 - Improper Integrals

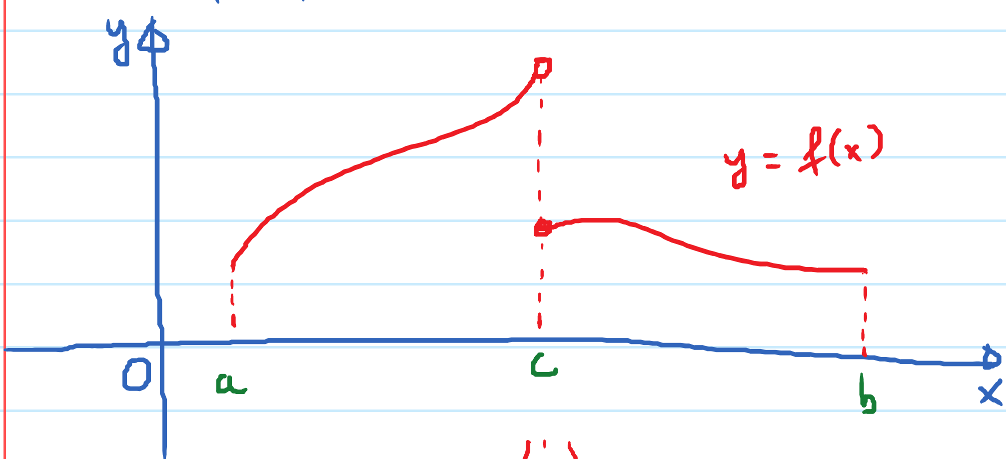
Monday, June 17, 2019 12:02 PM

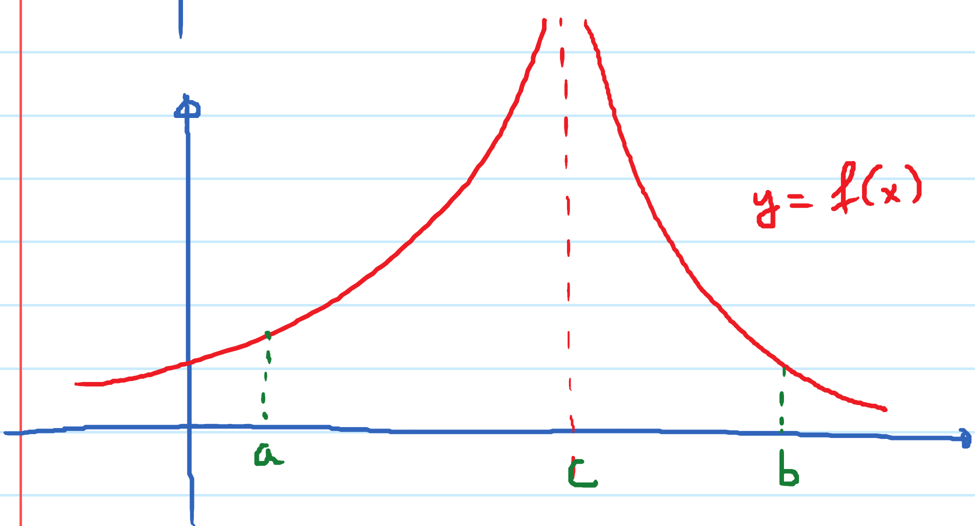
$$\int_a^b f(x)\,dx \underset{\text{F.T.C.}}{=} F(x)\Big|_a^b = F(b) - F(a)$$

Requirement of F.T.C.

1. f must be continuous on $[a, b]$
2. a, b are finite ; i.e., they are real #s.

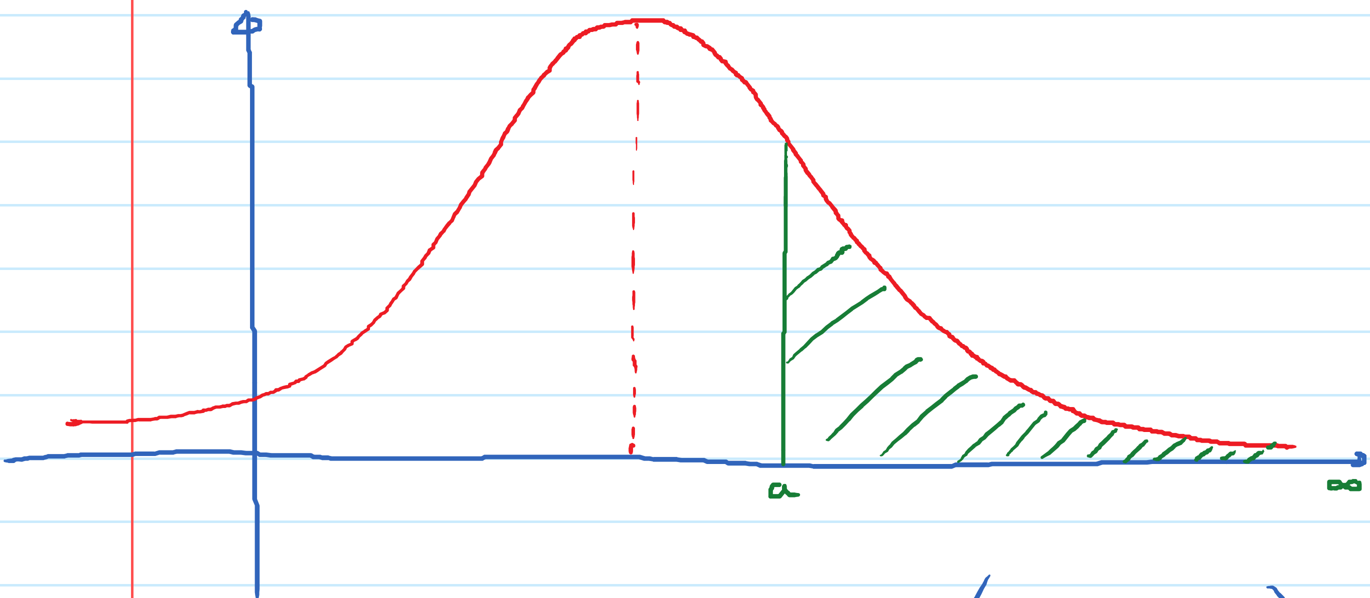
What if f has discontinuities?





Monday, June 17, 2019 12:06 PM

What if $a = \infty$ or $b = \infty$ or both



$\int_a^b f(x)\,dx$ is improper if $\begin{cases} \text{(Type I improper) } a = \infty \text{ or } b = \infty \text{ or both} \\ \text{(Type II improper) } f \text{ is not continuous on } [a,b]. \end{cases}$

Type I Improper Integrals:

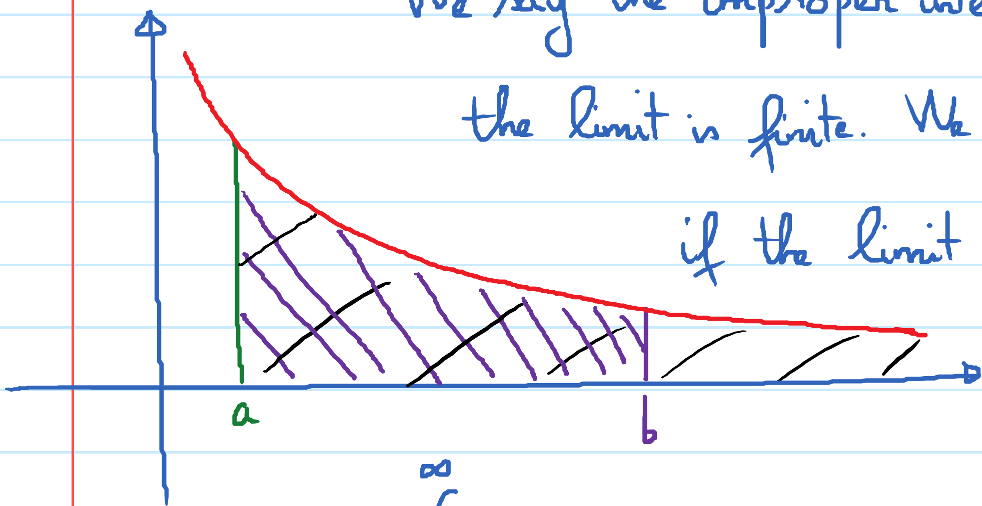
$$\int_a^{\infty} f(x)\,dx \ ; \quad \int_{-\infty}^{b} f(x)\,dx \ ; \quad \int_{-\infty}^{\infty} f(x)\,dx$$

(Assume f is continuous on interval of consideration.)

Monday, June 17, 2019 12:10 PM

variable

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^{\boxed{b}} f(x)\,dx$$



We say the improper integral converges if the limit is finite. We say it diverges if the limit is infinite or DNE

E.g. 1 ① $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

By formula:

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty} \boxed{\int_1^b \frac{1}{x^3}\,dx}$$

$$\int_1^b \frac{1}{x^3}\,dx = \int_1^b x^{-3}\,dx \underset{\text{FTC}}{=} \left(\frac{x^{-2}}{-2}\right)\Bigg|_1^b = \left(-\frac{1}{2x^2}\right)\Bigg|_{\boxed{1}}^{\boxed{b}}$$

$$= \left(-\frac{1}{2b^2}\right) - \left(-\frac{1}{2}\right) = \boxed{-\frac{1}{2b^2}+\frac{1}{2}}$$

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}$$

Conclusion: $\int_1^{\infty} \frac{1}{x^3}\,dx$ converges and it converges to $\frac{1}{2}$.

* $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\left(\int_{\boxed{a}}^{b} f(x)\,dx\right)$ variable

* $\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{c} f(x)\,dx}_{\text{into a limit}} + \underbrace{\int_{c}^{\infty} f(x)\,dx}_{\text{into a limit}}$

If both limits are finite, the integral converges.

If one of the limits (or both) is infinite or DNE, the integral diverges.

② $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

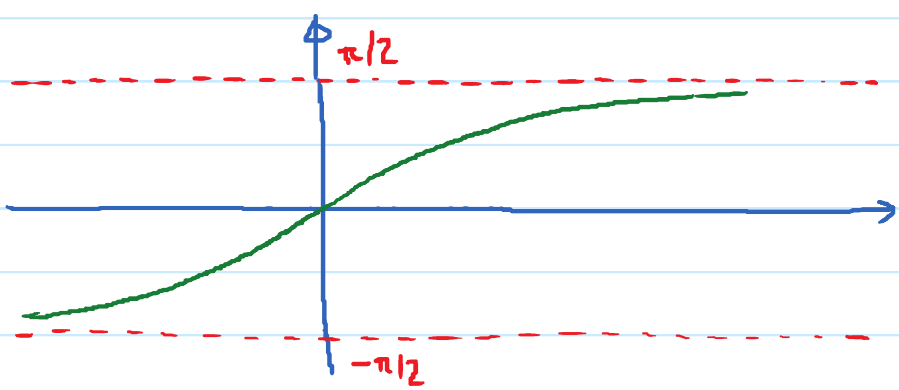
$$= \lim_{a \to -\infty} \left(\int_{a}^{0} \frac{1}{1+x^2}\,dx \right) + \lim_{b \to \infty} \left(\int_{0}^{b} \frac{1}{1+x^2}\,dx \right)$$

* Find $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x)$

* $$\lim_{a \to -\infty} \left(\arctan(x) \Big|_{a}^{0} \right) + \lim_{b \to \infty} \left(\arctan(x) \Big|_{0}^{b} \right)$$

* $$\lim_{a \to -\infty} \left(\cancel{\arctan(0)}^{\,0} - \arctan(a) \right) + \lim_{b \to \infty} \left(\arctan(b) - \cancel{\arctan(0)}^{\,0} \right)$$

* $$\underbrace{\lim_{a \to -\infty} \left(- \overbrace{\arctan(a)}^{-\pi/2} \right)}_{\pi/2} + \underbrace{\lim_{b \to \infty} \left(\arctan(b) \right)}_{\pi/2} = \pi.$$



Monday, June 17, 2019 12:34 PM

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi}$$

<u>E.g. 2</u> Type I improper integrals that diverge.

① $\displaystyle\int_{-\infty}^{\infty} (x^3 - 3x^2)\,dx$

$$= \int_{-\infty}^{0} (x^3 - 3x^2)\,dx + \int_{0}^{\infty} (x^3 - 3x^2)\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} (x^3 - 3x^2)\,dx + \lim_{b \to \infty} \int_{0}^{b} (x^3 - 3x^2)\,dx$$

$$= \lim_{a \to -\infty} \left[\left(\frac{x^4}{4} - x^3 \right) \Big|_{a}^{0} \right] + \lim_{b \to \infty} \left[\left(\frac{x^4}{4} - x^3 \right) \Big|_{0}^{b} \right]$$

$$= \underbrace{\lim_{a \to -\infty} \left[-\frac{a^4}{4} + a^3 \right]}_{-\infty} + \lim_{b \to \infty} \left[\frac{b^4}{4} - b^3 \right]$$

Since one of the limits is infinite (in this case actually both of them are infinite), the integral diverges.

2. $\displaystyle\int_{-\infty}^{0} x \cdot e^{-4x}\,dx$

$$= \lim_{a \to -\infty} \left[\int_{a}^{0} x \cdot e^{-4x}\,dx \right]$$

$*$ $\displaystyle\int x \cdot e^{-4x}\,dx \longrightarrow$ Integration by parts.

$$\begin{cases} u = x \\ dv = e^{-4x}\,dx \end{cases} \longrightarrow \begin{cases} du = dx \\ v = -\frac{1}{4} e^{-4x} \end{cases}$$

$$\int x \cdot e^{-4x}\,dx = -\frac{1}{4} x e^{-4x} + \frac{1}{4} \int e^{-4x}\,dx$$

$$= \lim_{a \to -\infty} \left[-\frac{1}{4} x e^{-4x} \Big|_a^0 + \frac{1}{4} \int_a^0 e^{-4x} \, dx \right]$$

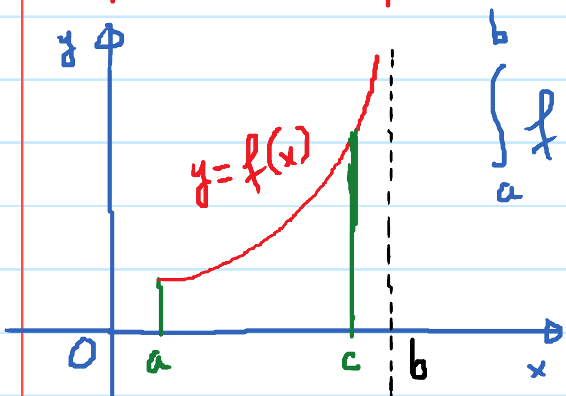
$$= \lim_{a \to -\infty} \left[\frac{1}{4} a e^{-4a} + \frac{1}{4} \cdot \left(-\frac{1}{4} e^{-4x} \right) \Big|_a^0 \right]$$

$$= \lim_{a \to -\infty} \left[\underbrace{\boxed{\frac{1}{4} a e^{-4a}}}_{\infty} - \frac{1}{16} + \underbrace{\frac{1}{16} e^{-4a}}_{\infty} \right] = \infty$$

$$\frac{1}{4} \lim_{a \to -\infty} a e^{-4a} = \frac{1}{4} \lim_{a \to -\infty} \frac{a}{e^{4a}} \underset{\text{L'Hopital}}{=} \frac{1}{4} \lim_{a \to -\infty} \frac{1}{4 e^{4a}}$$

So the integral diverges.

Type 2 Improper Integrals:



$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \left[\int_a^{\boxed{c}} f(x)\,dx \right]$$

variable