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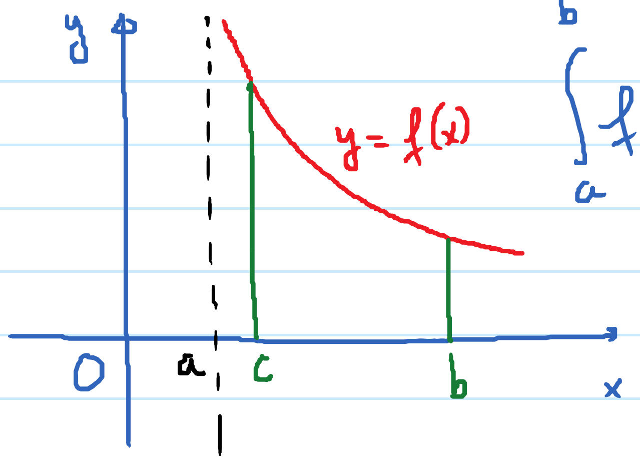
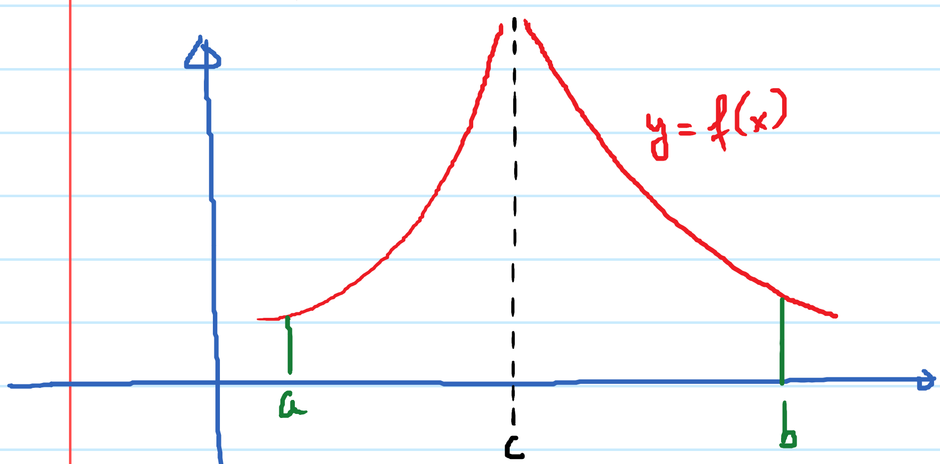



$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \left[ \int_{\boxed{c}}^{b} f(x)\,dx \right]$$

variable




$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{\text{turn into limit}} + \underbrace{\int_c^b f(x)\,dx}_{\text{turn into limit}}$$

If both limits are finite, integral converges

If either limit is infinite, integral diverges.

E.g.3.

$$\int_0^8 \frac{3}{\sqrt{8-x}}\,dx.$$

Improper of type II b/c it has a discontinuity at $x=8$.

$$\int_0^8 \frac{3}{\sqrt{8-x}}\,dx = \lim_{c\to 8^-}\boxed{\left[\int_0^c \frac{3}{\sqrt{8-x}}\,dx\right]}$$

$$\int_0^c \frac{3}{\sqrt{8-x}}\,dx = \int_0^c 3(8-x)^{-\frac{1}{2}}\,dx$$

$$= -6(8-x)^{\frac{1}{2}}\Big|_0^c$$

$$= -6(8-c)^{\frac{1}{2}} + 6(8)^{\frac{1}{2}}$$

$$= \boxed{-6\sqrt{8-c}+6\sqrt{8}}$$

$$\lim_{c\to 8^-}\left(-6\sqrt{8-c}+6\sqrt{8}\right) = \boxed{6\sqrt{8}}.$$

2. $$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

This is improper of Type II b/c it has a discontinuity at $x = 1$ in $[0, 9]$

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$= \lim_{c\to 1^-} \int_0^c \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{c\to 1^+} \int_c^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$\int \frac{1}{\sqrt[3]{x-1}}\,dx = \int (x-1)^{-\frac{1}{3}}\,dx = \frac{3}{2}(x-1)^{\frac{2}{3}}$$

$$= \lim_{c\to 1^-} \frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_0^c + \lim_{c\to 1^+} \frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_c^9$$

$$= \lim_{c\to 1^-} \left[\frac{3}{2}(c-1)^{\frac{2}{3}} - \frac{3}{2}\right] + \lim_{c\to 1^+} \left[ 6 - \frac{3}{2}(c-1)^{\frac{2}{3}}\right]$$

$$= -3/2 + 6 = \boxed{9/2}$$

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Eg. 4. (1) $\displaystyle\int_0^1 \frac{dx}{5x-3}$ → Type II improper b/c it has a discontinuity at $x=\frac{3}{5}$ within $[0,1]$

$$= \int_0^{\frac{3}{5}} \frac{dx}{5x-3} + \int_{\frac{3}{5}}^{1} \frac{dx}{5x-3}$$

$$= \lim_{c\to\frac{3}{5}^-} \boxed{\int_0^c \frac{dx}{5x-3}} + \lim_{c\to\frac{3}{5}^+} \boxed{\int_c^1 \frac{dx}{5x-3}}$$

$$\int \frac{dx}{5x-3} = \frac{1}{5}\ln|5x-3|$$

$$= \lim_{c\to\frac{3}{5}^-}\left[\frac{1}{5}\ln|5x-3|\,\Big|_0^c\right] + \lim_{c\to\frac{3}{5}^+}\left(\frac{1}{5}\ln|5x-3|\right)\Big|_c^1$$

$$= \underbrace{\lim_{c\to\frac{3}{5}^-}\left(\frac{1}{5}\underbrace{\ln|\overset{0}{5c-3}|}_{-\infty} - \frac{1}{5}\ln|3|\right)}_{-\infty} + \lim_{c\to\frac{3}{5}^+}\left(\frac{1}{5}\ln|2| - \frac{1}{5}\ln|5c-3|\right)$$

The integral diverges.

(2) $\displaystyle\int_0^1 \frac{e^{1/x}}{x^2}\,dx$ Improper b/c discontinuity at $x=0$

$$\int_0^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{e^{1/x}}{x^2}\,dx$$

$$\int \frac{e^{\overset{u}{1/x}}}{\underbrace{x^2}\,dx}_{-du}$$

Let $u = \frac{1}{x}$ ; $du = -\frac{1}{x^2}\,dx$

$$-\int e^u\,du = -e^u = \boxed{-e^{1/x}}$$

$$\lim_{c\to 0^+}\left(-e^{1/x}\right)\Big|_c^1$$

$$\lim_{c\to 0^+}\left(-e + e^{1/c}\right) = \infty$$

So, the integral diverges.

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## p-integrals:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{converges to } \dfrac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p \le 1. \end{cases}$$

## Why?

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

$$= \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty} \left(\frac{x^{-p+1}}{-p+1}\right)\Bigg|_1^b$$

$$= \lim_{b\to\infty} \left[\frac{(b)^{\boxed{-p+1}}}{-p+1} - \frac{1}{-p+1}\right]$$

If $p<1$. Integral diverges.

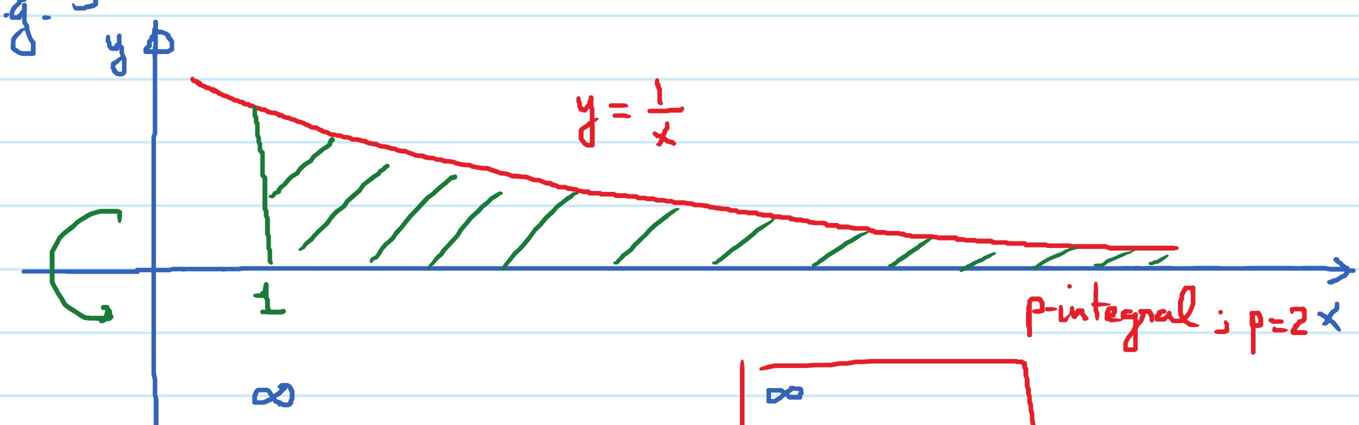
If $p > 1$, integral converges to: $-\dfrac{1}{-p+1} = \dfrac{1}{p-1}$.

If $p = 1$: $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \left(\ln|b|\right) = \infty$$

Eg. 5




p-integral; $p=2$

$$V = \pi \cdot \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \pi \cdot \boxed{\int_1^\infty \frac{1}{x^2}\,dx} = \boxed{\pi}.$$

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Surface area $S = 2\pi \int_1^\infty \frac{1}{x} \cdot \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx$

$$S = 2\pi \cdot \int_1^\infty \frac{1}{x} \underbrace{\boxed{\sqrt{1 + \frac{1}{x^4}}}}_{\geq 1} dx \rightarrow \geq \frac{1}{x}$$

$$S \geq 2\pi \cdot \int_1^\infty \frac{1}{x}\, dx \rightarrow \infty$$

So, $S$ is infinite.