Improper Integrals

Key formulas

Improper integrals of type 1: Integrate over an infinite interval

1. If f is a continuous function on $[a, \infty)$, then
$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$
2. If f is a continuous function on $(-\infty, b]$, then
$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$
3. If f is a continuous function on $(-\infty, \infty)$, then
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx,$$
where c is any real number.

Improper integrals of type 2: Integrate a discontinuous function

1. If f is a continuous function on $[a, b)$, and has an infinite discontinuity at b then
$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx.$$
2. If f is a continuous function on $(a, b]$, and has an infinite discontinuity at a then
$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx.$$
3. If f is a continuous function on $[a, b]$, and has an infinite discontinuity at c in (a, b)then
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

For both types of improper integrals: In case 1 and case 2, the improper integral **converges** if the limit exists, it **diverges** if the limit does not exist. In case 3, the improper integral on the left side **diverges** if **either** improper integral on the right side **diverges**. It **converges** if **both** improper integrals on the right side **converge**.
Important result about **p-integrals**
$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$

Example 1: Typer 1 improper integrals that converge

Evaluate the given integrals

1. $\int_1^{\infty} \frac{1}{x^3} dx$

2. $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Solution

Write the solution here

Example 2: Typer 1 improper integrals that diverge

Evaluate the given integrals

1. $\int_{-\infty}^{\infty} (x^3 - 3x^2) dx$

2. $\int_{-\infty}^{0} xe^{-4x} dx$

Solution

Write the solution here

Example 3: Typer 2 improper integrals that converge

Explain why the given integral is improper and evaluate it.

1. $\int_0^8 \frac{3}{\sqrt{8-x}}dx$

2. $\int_0^9 \frac{1}{\sqrt[3]{x-1}}dx$

Solution

Write the solution here

Example 4: Typer 2 improper integrals that diverge

Explain why the given integral is improper and evaluate it.

1. $\int_0^1 \frac{dx}{5x-3}$

2. $\int_0^1 \frac{e^{1/x}}{x^2}dx$

Solution

Write the solution here

Example 5: An application-Volume and surface area

Consider the solid formed by revolving the infinite region bounded by the graph of $y = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ about the x-axis. Use the result about p-integrals to show that this solid has a finite volume and an infinite surface area.

Solution

Write the solution here