# Series

## Key formulas

If we add the terms of an infinite sequence $\{a_n\}$, we get an **infinite series**

$$a_1 + a_2 + \ldots + a_n + \ldots.$$

The notation for an infinite series is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots.$$

The numbers $a_1$, $a_2$, etc. are called the terms of the series. The **mth partial sum** of the series, $S_m$, is the sum of the first $m$ terms of the series and is given by

$$S_m = a_1 + a_2 + \ldots + a_m = \sum_{n=1}^{m} a_n.$$

If the sequence of partial sums $\{S_m\}$ converges to a real number $s$, i.e., if $\lim_{m\to\infty} S_m = s$, then we say the series $\sum_{n=1}^{\infty} a_n$ **converges** to $s$. The number $s$ is the **sum** of the series and we write

$$\sum_{n=1}^{\infty} a_n = s.$$

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots.$$

Every term is obtained from the preceding term by multiplying it by the **common ratio** $r$ (the notation $\sum_{n=0}^{\infty} ar^n$ is also used for a geometric series).

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{diverges} & \text{if } |r| \geq 1 \\ \text{converges to } \frac{a}{1-r} & \text{if } -1 < r < 1 \end{cases}$$

**A telescoping series** such as $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ is another useful example of infinite series.

**nth term test for divergence:** If the series $\sum_{n=1}^{\infty} a_n$ converges, then the limit of the sequence of terms must be 0, i.e., $\lim_{n\to\infty} a_n = 0$.
Equivalently, if the limit of the sequence of terms of a series is nonzero or does not exist, the series diverges. In other words, if $\lim_{n\to\infty} a_n \neq 0$ or if $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

**Note:** The above test says nothing about convergence, i.e., if $\lim_{n\to\infty} a_n = 0$, we **cannot** conclude that $\sum_{n=1}^{\infty} a_n$ converges.

**Properties of infinite series:** If both the series $\sum a_n$ and $\sum b_n$ converge, then

1. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ for any constant $c$
2. $\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(a_n \pm b_n)$

## Example 1: Find partial sums

Find the sequence of partial sums $S_1, S_2, \ldots, S_6$ of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

## Solution

Write the solution here

## Example 2: Geometric series

Find the common ratio and determine whether the geometric series converges or diverges. Rewrite using summation notation. If the series converges, find the sum.

1. $3 - 4 + \dfrac{16}{3} - \dfrac{64}{9} + \ldots$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

## Solution

Write the solution here

### Example 3: Telescoping series

Find a formula for the **mth partial sum** of the series and use it to determine whether the series converges or diverges. If it converges, find the sum.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
2. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

### Solution

Write the solution here

### Example 4: nth-term test

Let $a_n = \dfrac{n}{2n+3}$.

1. Does the sequence $\{a_n\}$ converge? Why?
2. Does the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converge? Why?

### Solution

Write the solution here

### Example 5: Properties of series

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+(0.9)^n\right)$.

### Solution

Write the solution here

### Example 6: Make a series converge

Find all values of $x$ for which the series converges. Then write the sum of the series as a function of $x$.

1. $\displaystyle\sum_{n=1}^{\infty}(-5)^n x^n$
2. $\displaystyle\sum_{n=0}^{\infty}5\left(\frac{x-2}{3}\right)^n$

### Solution

Write the solution here