Series

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Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \dots\}$

An infinite series is obtained by adding all the terms of an infinite sequence:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Summation notation.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Sequence of partial sums of a series

$S_1 = a_1 \rightarrow$ 1st partial sum of the series

$S_2 = a_1 + a_2 \rightarrow$ 2nd partial sum of the series

$S_3 = a_1 + a_2 + a_3 \rightarrow$ 3rd partial sum

$\vdots$

$S_m = a_1 + a_2 + a_3 + \cdots + a_m \rightarrow$ mth partial sum

$\rightarrow$ sequence of partial sums: $S_1, S_2, S_3, \dots, S_m$

If the sequence of partial sums converges to a limit, i.e., if $\lim_{m \to \infty} S_m = s$, s is a real #, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges and we write

$$\sum_{n=1}^{\infty} a_n = \boxed{s} \longrightarrow \text{sum of the series}$$

If the sequence of partial sums diverges, then we say that the series: $\sum_{n=1}^{\infty} a_n$ diverges.

E.g. Series : $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$S_1 = \frac{1}{2}$; $S_2 = \frac{1}{2} + \frac{1}{4} = 0.75$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$

$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375$

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$$S_5 = S_4 + \frac{1}{32} = 0.96875$$

E.g.1: $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$S_1 , S_2 , S_3 , \dots , S_6$$

$$S_1 = 1 \; ; \quad S_2 = 1 + \frac{1}{2!} = 1 + \frac{1}{2} = 1.5$$

$$S_3 = 1 + \frac{1}{2!} + \frac{1}{3!} \dots$$

$$S_6 = 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{6!}$$

Special Series

* Geometric series:

E.g. $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

mult. by $\frac{1}{2}$ mult. by $\frac{1}{2}$ multiply by $\frac{1}{2}$

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In general, a geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

mult. by r multiply by r multi. by r

r is called the common ratio of the series.

Case 1. If $|r| \geq 1$, then the series diverges.

Case 2: If $|r| < 1$, then the series converges.

It converges to $s = \dfrac{\text{first term}}{1 - \text{common ratio}}$.

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } |r| < 1}$$

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E.g. 2

① $$3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$$

mult. by $-\frac{4}{3}$ mult. by $-\frac{4}{3}$ mult. by $-\frac{4}{3}$

$$\sum_{n=1}^{\infty} a r^{n-1} = \sum_{n=1}^{\infty} 3\left(-\frac{4}{3}\right)^{n-1}$$

Since the common ratio is $r = -\frac{4}{3}$ and $|r| > 1$, the series diverges.

② $$\sum_{n=1}^{\infty} \frac{e^{n}}{3^{n-1}} = \sum_{n=1}^{\infty} \underset{a}{e} \cdot \left(\underset{r}{\boxed{\frac{e}{3}}}\right)^{n-1}$$

$$\sum_{n=1}^{\infty} \frac{e^{n}}{3^{n-1}} = e + \frac{e^{2}}{3} + \frac{e^{3}}{9} + \frac{e^{4}}{27} + \frac{e^{5}}{81} + \cdots$$

$\cdot\frac{e}{3}$ $\cdot\frac{e}{3}$ $\cdot\frac{e}{3}$ $\cdot\frac{e}{3}$

Common Ratio $= r = \frac{e}{3}$, $|r| < 1$ → series converges

It converges to $\boxed{s = \frac{e}{1 - \frac{e}{3}}}$

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Telescoping Series

①
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right]$$

$$\left(n^{\text{th}} \text{ term: } a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \underbrace{\left(1 - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{a_2} + \underbrace{\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right)}_{a_3} + \cdots$$

Sequence of partial sums S_m.

$$S_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \left[\frac{1}{n} - \frac{1}{n+1} \right]$$

$$S_m = \underbrace{\left(1 - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{a_2} + \cdots + \underbrace{\left(\cancel{\frac{1}{m}} - \frac{1}{m+1}\right)}_{a_m}$$

So, $S_m = 1 - \dfrac{1}{m+1}$.

So, $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{m\to\infty} S_m = \lim_{m\to\infty} \left(1 - \frac{1}{m+1}\right) = \boxed{1}$$

②

$$\sum_{n=1}^{\infty} \underbrace{\ln\left(\frac{n}{n+1}\right)} = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \cdots$$

→ property of ln: $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$

$$\sum_{n=1}^{\infty} \left[\ln(n) - \ln(n+1)\right]$$

$$= \underbrace{\left[\ln(1) - \ln(2)\right]}_{a_1} + \underbrace{\left[\ln(2) - \ln(3)\right]}_{a_2} + \cdots$$

Sequence of partial sums:

$$S_m = \sum_{n=1}^{m} \left[\ln(n) - \ln(n+1)\right]$$

$$= \left[\ln(1) - \cancel{\ln(2)}\right] + \left[\cancel{\ln(2)} - \cancel{\ln(3)}\right] + \cdots + \left[\cancel{\ln(m)} - \ln(m+1)\right]$$

$$= \cancelto{0}{\ln(1)} - \ln(m+1)$$

$S_m = -\ln(m+1)$ → m^{th} partial sum

$$\lim_{m\to\infty} S_m = \lim_{m\to\infty}\left(-\ln(m+1)\right) = -\infty$$ → series diverges.

nth term test for divergence.

Given a series $\sum_{n=1}^{\infty} \boxed{a_n} \xrightarrow{} n^{\underline{th}}$ term.

If $\lim_{n\to\infty} a_n \neq 0$ or DNE, then the series diverges

Note. If $\lim_{n\to\infty} a_n = 0$, the test fails, i.e., we can draw no conclusion about convergence or divergence.

E.g. $a_n = \frac{n}{2n+3}$

The sequence $\{a_n\}$ converges b/c $\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$.

The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{2n+3}$ diverges b/c the nth term test. (i.e., $\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$)

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Properties of Series. $\sum a_n$: converges.

① $$\sum_{n=1}^{\infty} c\cdot a_n = c\sum_{n=1}^{\infty} a_n$$

E.g. $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\sum_{n=1}^{\infty} \frac{2019}{n(n+1)} = 2019\, \overbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}^{=1}$$

$$= 2019.$$

② $\sum a_n = s$; $\sum b_n = t$

$$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = s + t$$

E.g. $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \; ; \quad \sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \frac{e}{1-\frac{e}{3}}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \pm \frac{e^n}{3^{n-1}}\right) = 1 \pm \frac{e}{1-\frac{e}{3}}$$

Eg. 6

① $$\sum_{n=1}^{\infty} (-5)^n x^n = \sum_{n=1}^{\infty} (-5x)^n$$

This is a geometric series with common ratio $= -5x$

For it converge, $|r| < 1$

So, $|-5x| < 1$

$$5 \cdot |x| < 1 \rightarrow |x| < \frac{1}{5}$$

$$\rightarrow \boxed{-\frac{1}{5} < x < \frac{1}{5}}$$

② $$\sum_{n=0}^{\infty} 5\left(\frac{x-2}{3}\right)^n \rightarrow \text{geometric series}$$

Common ratio $r = \frac{x-2}{3}$.

For it to converge, $|r| < 1$.

$$\left|\frac{x-2}{3}\right| < 1 \rightarrow |x-2| < 3.$$

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(Recall:

$|\text{Stuff}| < a$

$\leftrightarrow -a < \text{Stuff} < a$)

$$-3 < x-2 < 3$$

$$\boxed{-1 < x < 5}$$