# The Integral Test

Wednesday, June 19, 2019 12:00 PM

Given a series $\sum_{n=1}^{\infty} a_n$ where $a_n$ is given by

$$a_n = \underbrace{f}_{\text{function}}(n)$$

**Assume:** The function $f$ is positive, continuous, decreasing on $[1, \infty)$ (or $[N, \infty)$ for some $N>0$)

The integral test says that:

If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

**Note:** The value $\int_1^{\infty} f(x)\,dx$ and the sum of the series $\sum_{n=1}^{\infty} a_n$ are different, in general.

E.g. 1 $\displaystyle\sum_{n=1}^{\infty} \boxed{\frac{n}{n^2+1}} \rightarrow a_n$

$a_n = \dfrac{n}{n^2+1}$. So, $f(x) = \dfrac{x}{x^2+1}$ on $[1, \infty)$

Requirements for integral test

1. $f$ is positive on $[1, \infty)$ ✓
2. $f$ is continuous on $[1, \infty)$ ✓ (Denom. $\neq 0$ on $[1, \infty)$)
3. $f$ is decreasing on $[1, \infty)$

$$f'(x) \underset{\text{quotient rule}}{=} \frac{1\cdot(x^2+1) - 2x\cdot x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2} \leq 0 \quad \text{on } [1, \infty)$$

So, $f$ is decreasing on $[1, \infty)$

Apply the integral test

$$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx$$

$$\int \frac{x}{x^2+1}\,dx \qquad \text{let } u = x^2+1 \,.\; du = 2x\,dx$$

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \boxed{\frac{1}{2}\ln|x^2+1|}$$

So,

$$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\ln|x^2+1| \Big|_1^b\right]$$

$$= \lim_{b\to\infty}\left(\frac{1}{2}\ln|b^2+1| - \frac{1}{2}\ln 2\right) = \infty.$$

So, the integral diverges.

The integral test says that $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

E.g. 2. Rewrite in summation notation:

$$\sum_{n=2}^{\infty} \boxed{\frac{\ln n}{n}} \longrightarrow a_n$$

$a_n = \dfrac{\ln(n)}{n}$. So, $f(x) = \dfrac{\ln(x)}{x}$ on $[2, \infty)$

Requirements:

① $f$ is positive on $[2, \infty)$ ✓

② $f$ is continuous on $[2, \infty)$ ✓

③ $f$ is decreasing on $[2, \infty)$.

$$f'(x) = \frac{\frac{1}{x} \cdot x - 1 \cdot \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2} \leq 0$$

on $(3, \infty)$

So, $f$ is decreasing on $(3, \infty)$

$$\int_2^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{b \to \infty} \int_2^{b} \frac{\ln(x)}{x}\,dx$$

$$\lim_{b\to\infty} \left.\frac{[\ln(x)]^2}{2}\right|_2^b$$

$$\lim_{b\to\infty}\left[\left(\frac{[\ln(b)]^2}{2} - \frac{[\ln(2)]^2}{2}\right]\right. = \infty$$

So, the integral diverges. So does the series.

E.g. 3 ① $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$

$f(n) = \dfrac{\cos(\pi n)}{\sqrt{n}}$ is NOT positive on $[1, \infty)$

② $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$

The function is not decreasing.

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p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} p > 1 : \text{ the series converges} \\ p \leq 1 : \text{ the series diverges.} \end{cases}$$

For e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ ; $p = 1 \rightarrow$ series diverges.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ ; $p = 2 \rightarrow$ series converges.

Why ?

Wednesday, June 19, 2019 12:33 PM

Consider: $f(x) = \dfrac{1}{x^p}$ on $[1, \infty)$ — positive, cont., decreasing

$$\int_1^\infty \frac{1}{x^p}\,dx$$ — from result on p-integrals — converges if $p > 1$; diverges if $p \leq 1$

So does the series.

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Wednesday, June 19, 2019 12:36 PM

E.g ① $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}} \rightarrow p = \pi > 1 \rightarrow$ converges

② $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}} \rightarrow p = \frac{2}{3} < 1$

$\rightarrow$ diverges.