The comparison tests

Key formulas

Direct Comparison Test: Let $\sum a_n$ and $\sum b_n$ be series with **positive** terms and $0 < a_n \leq b_n$ for all $n \geq 1$ (or $n \geq N$ for some positive integer $N \geq 1$), then

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ converges.

In short, if the "larger" series converges, then the "smaller" series must converge. If the "smaller" series diverges, then the "larger" series must diverge.

Limit Comparison Test: Let $\sum a_n$ and $\sum b_n$ be series with **positive** terms, i.e., $a_n > 0$ and $b_n > 0$ for all $n \geq 1$ (or $n \geq N$ for some positive integer $N \geq 1$). If the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where L is a **finite** and **positive** number, then either both series converge of both series diverge.

Example 1: Using direct comparison test - compare with geometric series

Determine whether the series $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges or diverges.

Solution

Write the solution here

Example 2: Using direct comparison test - compare with p-series

Determine whether the series converges or diverges.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$

Solution

Write the solution here

Example 3: Using limit comparison test

The series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

1. A calculus student says that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} < \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$. And since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the given series diverges. Explain why this logic is wrong.
2. Apply the limit comparison test to show that the given series diverges.

Solution

Write the solution here

Example 4: Using limit comparison test

Determine whether the series converges or diverges: $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$.

Solution

Write the solution here

Example 5: Using limit comparison test

Determine whether the series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$.

Solution

Write the solution here