Comparison Tests.

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Recall: * n^{th} term test: $\sum_{n=1}^{\infty} a_n$.

$\lim_{n\to\infty} a_n \neq 0$ or DNE $\rightarrow$ series diverges.

* Integral test $\sum_{n=1}^{\infty} a_n$ where $a_n = f(n)$

Requirements for f: ① f positive
② f continuous
③ f decreasing
on $[1,\infty)$

$\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} a_n$ behave in the same way

Direct Comparison Test:

2 series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

Requirement: $a_n \geq 0$ and $b_n \geq 0$ for all n.
(Series must have all terms nonnegative)

If $a_n \leq b_n$ for all n then:

* If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
* If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

E.g. 1. $$\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$$

$$3 + 10^n > 10^n \text{ for all } n \geq 1$$

$$\frac{1}{3+10^n} < \frac{1}{10^n}$$

$$\underbrace{\boxed{\frac{9^n}{3+10^n}}}_{a_n} < \underbrace{\boxed{\frac{9^n}{10^n}}}_{b_n} \text{ for all } n \geq 1$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{9^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$. This series converges because it is a geometric series with common ratio $r = \frac{9}{10}$ so $|r| < 1$.

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converges

Since $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n} < \sum_{n=1}^{\infty} \frac{9^n}{10^n}$, the comparison test says that

$$\sum_{n=1}^{\infty} \frac{9^n}{3+10^n} \text{ converges.}$$

diverge

E.g.2

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ compare with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ $p = \frac{1}{2}$

$\sqrt{n} - 1 < \sqrt{n}$ for all $n \geq 2$.

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

Since the second series diverges (p-series where $p = \frac{1}{2} < 1$), the first series diverges.

② $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ compare with $\displaystyle\sum_{n=1}^{\infty} \frac{\cancel{n}}{n^{\cancel{2}}\sqrt{n}} = \boxed{\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}}$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$p = \frac{3}{2}$.

$$n-1 < n.$$

$$\frac{n-1}{n^2\sqrt{n}} < \frac{n}{n^2\sqrt{n}}$$

$$\frac{n-1}{n^2\sqrt{n}} < \frac{1}{n\sqrt{n}}.$$

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

converges

The second series converges b/c it is a p-series with $p = 3/2 > 1$. So, the first series converges.

E.g. 3

1 The solution shows that the given series $<$ a divergent series. This does not tell us anything about the original series.

$\longrightarrow$ Limit Comparison Test

Given 2 series: $\sum_{n=1}^{\infty} a_n$; $\sum_{n=1}^{\infty} b_n$; terms are all positive.

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L.$$

If L is a finite and positive number $(L > 0)$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ behave in the exact same way.

Back to problem

$$\sum_{n=1}^{\infty} \underbrace{\boxed{\frac{n}{n^2+1}}}_{a_n} \quad ; \text{ compare with } \sum_{n=1}^{\infty} \frac{n}{n^2}$$

$$= \sum_{n=1}^{\infty} \underbrace{\boxed{\frac{1}{n}}}_{b_n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+1} = \boxed{1}$$ positive; finite.

The limit comparison test says that $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ behave the same way.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series $p = 1$), the original series diverges.

E.g. 4 $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$ a_n

limit comparing it to $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n^2}$ b_n

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n+2}}{2n^2+n+1} \cdot \frac{2n^2}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{2n^2\sqrt{n+2}}{(2n^2+n+1)\sqrt{n}} = 1$$

So, the given series and $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n^2}$ behave in the same way.

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{2n^2} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$$

p-series with $p = \frac{3}{2} > 1$. So it converges.

Therefore, the original series converges

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E.g. 5. $\displaystyle\sum_{n=1}^{\infty} \boxed{\frac{n2^n}{4n^3+1}}$ a_n

Limit comparing it to $\displaystyle\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{4n^3} = \sum_{n=1}^{\infty} \boxed{\frac{2^n}{4n^2}}$ b_n

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n \cdot \cancel{2^n}}{4n^3+1} \cdot \frac{4n^2}{\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{4n^3}{4n^3+1} = 1$$

So, the original series and the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2}$ behave in the same way.

Since $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2}$ diverges b/c of the nth term test, the given series must diverge.