Alternating Series Test

Monday, June 24, 2019 12:10 PM

An alternating series is a series of form

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

Here: $a_n > 0$.

A.S.T: If a_n satisfies:

① $\lim_{n\to\infty} a_n = 0$ ② $a_{n+1} \leq a_n$: i.e. a_n is non-increasing

Then the series converges.

E.g 1: $\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\boxed{\frac{1}{n}}}_{a_n}$. converges by the A.S.T

Conditions for A.S.T.

$a_n = \frac{1}{n}$.

① $\lim_{n\to\infty} a_n = 0$. ✓ ② $a_{n+1} \leq a_n$ ✓

$\lim_{n\to\infty} \frac{1}{n} = 0$ $\frac{1}{n+1} \leq \frac{1}{n}$.

So, the A.S.T. applies.

(2) $\sum_{n=1}^{\infty} (-1)^{n+1} \boxed{\frac{n}{e^n}}$. → alternating Series.

a_n

2 requirements.

$$a_n = \frac{n}{e^n}.$$

(1) $\lim_{n\to\infty} a_n = 0$: ✓

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

(2) $a_{n+1} \leq a_n$: ✓

$\frac{n+1}{e^{n+1}} \leq \frac{n}{e^n}$ (not immediately clear why this is true)

→ take derivative of $f(n) = a_n$ and verify that $f'(n) \leq 0$, hence, a_n is non-increasing.

$$f(n) = \frac{n}{e^n} \to f'(n) = \frac{1\cdot e^n - e^n \cdot n}{e^{2n}}$$

$$f'(n) = \frac{e^n(1-n)}{e^{2n}} = \frac{1-n}{e^n} \leq 0$$

So, a_n is non-increasing.

Hence, the A.S.T applies and we can conclude that the series converges.

E.g.2.

① A.S.T does not apply to $\sum_{n=1}^{\infty} (-1)^n \boxed{\frac{5n-1}{4n+1}}$ a_n

b/c $\lim_{n\to\infty} a_n = \frac{5}{4} \neq 0$.

Divergence Test tells us that the series diverges.

② A.S.T does not apply to $\sum_{n=1}^{\infty} (-1)^{n+1} \sqrt{n}$ a_n

b/c $\lim_{n\to\infty} a_n = \infty \neq 0$.

Divergence test tells us that the series diverges

E.g. 3.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \text{ converges by A.S.T.}$$

why? $a_n = \frac{1}{n^2}$

(1) $\lim_{n\to\infty} a_n = 0$ ✓

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

(2) $a_{n+1} \leq a_n$ ✓

$$\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$$

(2) Suppose this converges to s.

The problem asks us to find how many terms (N) to use so that

$$|S_N - s| \leq 0.001$$

where

$$S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2}$$

A.S. Theorem says that:

$$|S_N - s| \leq a_{N+1}$$

absolute value of the first neglected term

Since we want $|S_N - s| \leq 0.001$, it suffices to require that

$$a_{N+1} \leq 0.001$$

$$\longleftrightarrow \frac{1}{(N+1)^2} \leq 0.001 \qquad \left(\text{Recall } a_n = \frac{1}{n^2}\right)$$

$$\longleftrightarrow 1 \leq 0.001(N+1)^2$$

$$\longleftrightarrow \frac{1}{0.001} \leq (N+1)^2$$

$$\longleftrightarrow 1000 \leq (N+1)^2$$

$$\longleftrightarrow \sqrt{1000} \leq N+1$$

$$\sqrt{1000} - 1 \leq N$$

$$N \geq 30.62$$

If we use 31 terms, we can achieve the level of accuracy that we want.

Absolute convergence series v.s. conditional convergence series.

A series $\sum a_n$ converges absolutely if

① $\sum a_n$ converges. (original series converges)

② $\sum |a_n|$ converges (abs. value series converges.)

A series $\sum a_n$ converges conditionally if

① $\sum a_n$ converges (original series converges)

② $\sum |a_n|$ diverges. (abs. value series div.)

E.g. of a series that converges conditionally.

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}}$$ converges by A.S.T.

Absolute value series: $\displaystyle\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n} \right|$

$\displaystyle = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series.

So: original converges but abs. diverges.

An e.g. of a series that converges absolutely.

$\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$. converges by A.S.T

Abs. value series: $\displaystyle\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right|$

$\displaystyle = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series

$(p=2>1)$

original and abs. converges

Monday, June 24, 2019 12:51 PM

E.g. 4.

① $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\sqrt{n}}$ ② $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n^4+1}}$

Both series converges by A.S.T.

Abs. value series:

① $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ ② $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+1}}$

$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

p-series with

$p = \frac{3}{2} > 1$. So converges

converges.

Reason: limit comparing it with $\sum \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{\sqrt{n^4+1}}} = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^2} = 1$$

② $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n\ln n}$. converges by AST

Abs. value series: $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges.

$f(x)=\frac{1}{x\ln(x)}$ on $[2,\infty)$

① Positive: ✓ ② Continuous: ✓

③ Decreasing ✓

$$\int_2^{\infty}\frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln(x)}\,dx$$

$u=\ln(x)$

$du=\frac{1}{x}dx$

$$\int \frac{1}{u}\,du = \ln(u) = \boxed{\ln(\ln(x))}$$

$$= \lim_{b\to\infty} \ln(\ln(x))\Big|_2^b$$

$$= \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(2)) = \infty$$

Monday, June 24, 2019 1:04 PM

Integral diverges. Hence, series diverges.

Original series converges. $\Big\}$ converges conditionally.
Abs. series diverges