The Ratio Test and The Root Test

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The Ratio Test

Series: $\sum_{n=1}^{\infty} a_n$

Calculate $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

1. If $L < 1$, then the series converges absolutely.

2. If $L > 1$ or $L = \infty$, then the series diverges.

3. If $L = 1$, then the test fails, we cannot draw any conclusion about convergence or divergence of series.

E.g. 1

1. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$.

$$a_n = (-1)^n \frac{n^3}{3^n}$$

$$a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\cancel{(-1)^{n+1}} (n+1)^3}{3^{n+1}} \cdot \frac{3^n}{\cancel{(-1)^n}\, n^3} \right| = \frac{(n+1)^3}{3n^3}$$

$$\lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \frac{1}{3} < 1$$

Ratio test says that the series converges absolutely.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$a_n = \frac{n^n}{n!}$$

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{\boxed{(n+1)!}} \cdot \frac{\boxed{\cancel{n!}}}{n^n} = \frac{(n+1)^{n+1}}{(n+1)^1 \cdot n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$$

Hence, the Ratio test says that the series diverges.

E.g. 2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

$$a_n = (-1)^n \frac{\sqrt{n}}{n+1}$$

$$a_{n+1} = (-1)^{n+1} \frac{\sqrt{n+1}}{n+2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\cancel{(-1)^{n+1}}\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\cancel{(-1)^n}\sqrt{n}} \right|$$

$$= \frac{\left(\sqrt{n+1}\right)(n+1)}{(n+2)\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{(n+1)\sqrt{n+1}}{(n+2)\sqrt{n}} = 1 \rightarrow \text{Test fails.}$$

$$\sum_{n=1}^{\infty} (-1)^n \cdot \boxed{\frac{\sqrt{n}}{n+1}} \rightarrow a_n = \frac{\sqrt{n}}{n+1}$$

① $\displaystyle\lim_{n\to\infty} a_n = 0$: $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$

② a_n non-increasing. ✓

$$f(n) = \frac{\sqrt{n}}{n+1} ; \quad f'(n) = \frac{\boxed{\frac{1}{2\sqrt{n}}(n+1) - \sqrt{n}}}{(n+1)^2}$$

$$= \frac{(n+1) - 2n}{2\sqrt{n}\,(n+1)^2} = \frac{1-n}{2\sqrt{n}(n+1)^2} \leq 0$$

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So, $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$ converges by A.S.T.

Abs. value series: $\sum_{n=1}^{\infty} \boxed{\frac{\sqrt{n}}{n+1}}$

Limit comparing this series with: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \boxed{\frac{1}{\sqrt{n}}}$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

So, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ behaves like $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

diverges b/c
p-series $p = \frac{1}{2}$.

So, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ diverge.

Final conclusion: $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$ converges conditionally.

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Root Test $\sum a_n$

Compute: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$

Same as Ratio Test after this.

E.g. 3.

① $\sum_{n=1}^{\infty} \boxed{\left(\frac{2n}{n+1}\right)^n} \longrightarrow a_n$

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{2n}{n+1}\right)^{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n}{n+1} = 2.$$

Series diverges.

② $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{-2}{n}\right)^n$

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(\frac{2}{n}\right)^{n}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{2}{n}\right) = 0$$

Series converges absolutely.

E.g. 4.

① $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ — $a_n = \frac{x^n}{n!}$, $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$$ regardless of the values of x.

The ratio test says that the series converges absolutely.

Conclusion: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all values of x.

② $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)}$

$$a_n = \frac{n^2 x^n}{2[1\cdot 2\cdot 3\cdots n]} = \frac{n^2 x^n}{2(n!)}$$

$$a_{n+1} = \frac{(n+1)^2 x^{n+1}}{2[(n+1)!]}$$

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$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 x^{n+1}}{\cancel{2}[(n+1)!]} \cdot \frac{\cancel{2}(n!)}{n^2 x^n}\right|$$

$$= \frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)} \cdot n^2} |x| = \frac{n+1}{n^2} |x|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n^2} |x| = 0$$

Converges for all values of x.