Taylor Polynomial.

Tuesday, June 25, 2019 12:02 PM

n^{th} Taylor Polynomial for a function f centered at α is given by the formula:

$$\boxed{T_n(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \frac{f'''(\alpha)}{3!}(x-\alpha)^3 + \cdots + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n}$$

If the center $\alpha = 0$, the n^{th} Taylor polynomial for f centered at 0 is called the nth Maclaurin polynomial:

$$\boxed{T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n}$$

Tuesday, June 25, 2019 12:15 PM

E.g.1 $f(x) = e^x$. Find Maclaurin polynomial T_1, T_2, T_3 for f. (center $x=0$)

$T_1(x) = f(0) + f'(0)\cdot x$; $f(0) = 1$

$f'(x) = e^x$; $f'(0) = 1$

$\boxed{T_1(x) = 1 + x}$ → 1^{st} Maclaurin polynomial

$T_2(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2$

$f''(x) = e^x$; $f''(0) = 1$

$\boxed{T_2(x) = 1 + x + \frac{1}{2}x^2}$ → 2^{nd} Maclaurin polynomial

$T_3(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$\boxed{T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$ → 3^{rd} Maclaurin polynomial.

$\boxed{T_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n}$ n^{th} M. poly.

E.g.2 T_1, T_2, T_3, T_4 (centered at $a = 1$) for $f(x) = \ln x$

	Plug in $a = 1$
$f(x) = \ln x$	$f(1) = \ln(1) = 0$
$f'(x) = \frac{1}{x}$	$f'(1) = 1$
$f''(x) = -\frac{1}{x^2}$	$f''(1) = -1$
$f'''(x) = \frac{2}{x^3}$	$f'''(1) = 2$
$f^{(4)}(x) = -\frac{6}{x^4}$	$f^{(4)}(1) = -6$

$$T_1(x) = f(1) + f'(1)(x-1) = \boxed{x - 1}$$

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$= \boxed{(x-1) - \frac{1}{2}(x-1)^2}$$

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= \boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}$$

$$T_4(x) = \boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4}$$

$T_1(1.1)$; $T_2(1.1)$; $T_3(1.1)$; $T_4(1.1)$ provide more and more accurate approx. to $\ln(1.1)$.

E.g. 3.

$$f(x) = \cos x$$

$$T_1(x) = 1$$

$$T_2(x) = 1 - \frac{1}{2}x^2$$

$$T_3(x) = 1 - \frac{1}{2}x^2$$

$$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$T_n(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots + (-1)^K \frac{x^{2K}}{(2K)!}$$

$$(n = 2K)$$

* $f(x) = \sin x$

$$T_1(x) = x \;;\; T_2(x) = x \;;\; T_3(x) = x - \frac{1}{6}x^3$$

$$T_4(x) = x - \frac{1}{6}x^3$$

Tuesday, June 25, 2019 12:50 PM

Taylor's Theorem.

Function $f(x)$

Use $n^{\underline{th}}$ Taylor polynomial centered at α to approximate $f(x)$: $T_n(x)$

nth remainder: $\left| \underbrace{f(x) - T_n(x)}_{R_n(x)} \right|$

$$\boxed{|R_n(x)| \leq M \cdot \frac{|x-\alpha|^{n+1}}{(n+1)!}}$$

where M = max. value of the $(n+1)$-derivative of the function on $[x, \alpha]$ or $[\alpha, x]$

1) $f(x) = \cos(x)$

$T_4(0.1)$ to approximate $\cos(0.1)$

($T_4(x)$: 4^{th} Maclaurin poly. for $f(x)$)

How good is this approximation?

Taylor's Theorem:

$$|R_4(x)| \leq M \cdot \frac{|x-a|^5}{5!}$$

$x = 0.1$; $a = 0$

M = max. of the $|5^{\text{th}}$ derivative$|$ of $f(x) = \cos x$

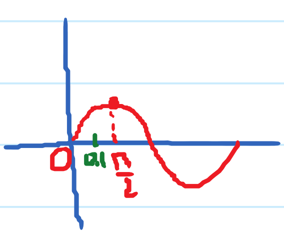
on $[0, 0.1]$

$f(x) = \cos x$; $f'(x) = -\sin x$; $f''(x) = -\cos x$

$f'''(x) = \sin x$; $f^{(4)}(x) = \cos x$; $f^{(5)}(x) = -\sin x$

$|f^{(5)}(x)| = |-\sin x| = |\sin x|$ on $[0, 0.1]$

$M = |\sin(0.1)|$



$$|R_4(x)| \leq |\sin(0.1)| \frac{(0.1)^5}{5!}$$

2

$T_5(1)$ to estimate e^1

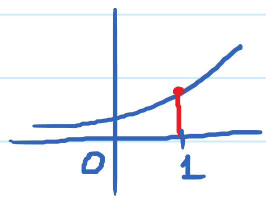
$T_5(x)$: 5^{th} Maclaurin poly. for $f(x) = e^x$.

$$|R_5(x)| \leq M \cdot \frac{|x-\alpha|^6}{6!}$$

$x = 1$; $\alpha = 0$

$M =$ max of $|f^{(6)}(x)|$ on $[0, 1]$

$f^{(6)}(x) = e^x$.



$M = e$

$$|R_5(x)| \leq e \cdot \frac{1^6}{6!} = \boxed{\frac{e}{720}}$$