Taylor Polynomials and Approximations

Key formulas

The **nth Taylor polynomial for a function f centered at** α is a polynomial of degree *n* denoted by $T_n(x)$ and it is given by the formula

$$T_n(x) = f(\alpha) + f'(\alpha)(x - \alpha) + \frac{f''(\alpha)}{2!}(x - \alpha)^2 + \dots + \frac{f^{(n)}(\alpha)}{n!}(x - \alpha)^n.$$

If the center $\alpha = 0$, then the nth Taylor polynomial for f centered at 0 is called the **nth Maclaurin polynomial** for f. So, the nth Maclaurin polynomial for f is given by

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

Remainder Estimate: Suppose that we use the value $T_n(x)$ of the nth Taylor polynomial centered at α to approximate the function value f(x) for some function f which is differentiable through order n + 1. The **nth** remainder $R_n(x)$ is defined as the difference between the value of the function at x and the value of the Taylor polynomial at x

$$R_n(x) = f(x) - T_n(x).$$

Taylor's theorem says that

$$|R_n(x)| \le M \cdot \frac{|x-\alpha|^{n+1}}{(n+1)!},$$

where M is the maximum value of $|f^{(n+1)}|$, the absolute value of the (n + 1)-derivative of f, on the interval $[x, \alpha]$. This provides us with an upper bound for the error of our approximation. **Note:** More precisely, Taylor's theorem says that there is a value z in $[x, \alpha]$ such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-\alpha)^{n+1}.$$

As a result,

$$f(x) = f(\alpha) + f'(\alpha)(x - \alpha) + \frac{f''(\alpha)}{2!}(x - \alpha)^2 + \dots + \frac{f^{(n)}(\alpha)}{n!}(x - \alpha)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x - \alpha)^{n+1}, \text{ for some } z \text{ in } [x, \alpha],$$

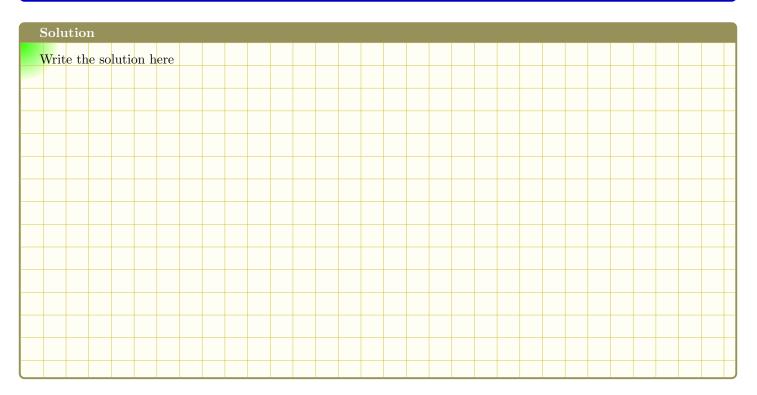
although it is often impossible to find the value of z.

Example 1: Maclaurin polynomial for $f(x) = e^x$

Find the Maclaurin polynomial T_1 , T_2 , T_3 for $f(x) = e^x$ and use the pattern to find the formula for the general nth degree Maclaurin polynomial for $f(x) = e^x$.

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Example 2: Taylor polynomial for $f(x) = \ln x$
Find the Taylor polynomial T_1, T_2, T_3, T_4 for $f(x) = \ln x$ centered at $\alpha = 1$ and use it to estimate $\ln(1.1)$



Example 3: Macl	aurin polyno	omial for $f($	$f(x) = \sin x, \ f(x) = \cos x$		
Find the Maclaurin	polynomials '	T_1, T_2, T_3, T_4	$_{4}$ for the function and use	$T_n(0.1)$ to estimate $f(0.1)$	
$1. \ f(x) = \cos x,$	use T_4 to estim	nate.	$2. \ f(x) = \sin x$	In x , use T_3 to estimate.	

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Example 4: Remainder estimate
Use Taylor's theorem to find the bound for the remainder when the Maclaurin polynomial $T_n(x)$ is used to estimate
f(x).
1. $T_4(0.1)$ for $f(x) = \cos x$ is used to estimate $\cos(0.1)$. 2. $T_5(1)$ for $f(x) = e^x$ is used to estimate e.

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