Taylor Polynomials and Approximations

Key formulas

The **nth Taylor polynomial for a function f centered at** α is a polynomial of degree n denoted by $T_n(x)$ and it is given by the formula

$$T_n(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n.$$

If the center $\alpha = 0$, then the nth Taylor polynomial for f centered at 0 is called the **nth Maclaurin polynomial for f**. So, the nth Maclaurin polynomial for f is given by

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Remainder Estimate: Suppose that we use the value $T_n(x)$ of the nth Taylor polynomial centered at α to approximate the function value $f(x)$ for some function f which is differentiable through order $n+1$. The **nth** remainder $R_n(x)$ is defined as the difference between the value of the function at x and the value of the Taylor polynomial at x

$$R_n(x) = f(x) - T_n(x).$$

Taylor's theorem says that

$$|R_n(x)| \leq M \cdot \frac{|x-\alpha|^{n+1}}{(n+1)!},$$

where M is the maximum value of $|f^{(n+1)}|$, the absolute value of the $(n+1)$-derivative of f, on the interval $[x, \alpha]$. This provides us with an upper bound for the error of our approximation.
Note: More precisely, Taylor's theorem says that there is a value z in $[x, \alpha]$ such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-\alpha)^{n+1}.$$

As a result,

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x-\alpha)^{n+1}, \text{ for some } z \text{ in } [x, \alpha],$$

although it is often impossible to find the value of z.

Example 1: Maclaurin polynomial for $f(x) = e^x$

Find the Maclaurin polynomial T_1, T_2, T_3 for $f(x) = e^x$ and use the pattern to find the formula for the general nth degree Maclaurin polynomial for $f(x) = e^x$.

Solution

Write the solution here

Example 2: Taylor polynomial for $f(x) = \ln x$

Find the Taylor polynomial T_1, T_2, T_3, T_4 for $f(x) = \ln x$ centered at $\alpha = 1$ and use it to estimate $\ln(1.1)$

Solution

Write the solution here

Example 3: Maclaurin polynomial for $f(x) = \sin x$, $f(x) = \cos x$

Find the Maclaurin polynomials T_1, T_2, T_3, T_4 for the function and use $T_n(0.1)$ to estimate $f(0.1)$

1. $f(x) = \cos x$, use T_4 to estimate.
2. $f(x) = \sin x$, use T_3 to estimate.

Solution

Write the solution here

Example 4: Remainder estimate

Use Taylor's theorem to find the bound for the remainder when the Maclaurin polynomial $T_n(x)$ is used to estimate $f(x)$.

1. $T_4(0.1)$ for $f(x) = \cos x$ is used to estimate $\cos(0.1)$.
2. $T_5(1)$ for $f(x) = e^x$ is used to estimate e.

Solution

Write the solution here