Power Series

Key formulas

A **power series centered at** α with **coefficients** $c_0, c_1, c_2, \dots$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-\alpha)^n = c_0 + c_1(x-\alpha) + c_2(x-\alpha)^2 + \dots + c_n(x-\alpha)^n + \dots$$

If the center $\alpha = 0$, then the power series centered at 0 has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

For a given power series $\sum_{n=0}^{\infty} c_n(x-\alpha)^n$, there are exactly three possibilities:

1. The series converges only at $x = \alpha$.
2. The series converges absolutely for all real number x.
3. The series converges for x in an interval centered at α. More precisely, there is a number $R > 0$ such that the series converges absolutely for all x in the interval $(\alpha - R, \alpha + R)$ and diverges for x in $(-\infty, \alpha - R) \cup (\alpha + R, \infty)$.

The number R in case 3 is called the **radius of convergence** of the series. (The radius of convergence is 0 in case 1 and ∞ in case 2.)
The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case 1, the interval of convergence is a single point $\{\alpha\}$. In case 2, the interval of convergence is $(-\infty, \infty)$. In case 3, the interval of convergence includes the interval $(\alpha - R, \alpha + R)$. However, the above result does not say anything about the endpoints $x = \alpha + R$ and $x = \alpha - R$. Anything can happen at these 2 endpoints in this case and hence they must be tested separately for convergence or divergence. Thus, in case 3 there are 4 possibilities for the interval of convergence:

$$(\alpha - R, \alpha + R); \quad [\alpha - R, \alpha + R); \quad (\alpha - R, \alpha + R]; \quad [\alpha - R, \alpha + R].$$

If a power series $\sum c_n(x-\alpha)^n$ has radius of convergence $R > 0$, then for x in the interval of convergence of the series, we can define a function

$$f(x) = c_0 + c_1(x-\alpha) + c_2(x-\alpha)^2 + c_3(x-\alpha)^3 + \dots = \sum_{n=0}^{\infty} c_n(x-\alpha)^n.$$

The function f is continuous and differentiable on the interval of convergence and we can find its derivative and its antiderivative by **term-by-term** differentiation and integration. More specifically,

$$f'(x) = c_1 + 2c_2(x-\alpha) + 3c_3(x-\alpha)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x-\alpha)^{n-1},$$

and

$$\int f(x)dx = C + c_0(x-\alpha) + c_1\frac{(x-\alpha)^2}{2} + c_2\frac{(x-\alpha)^3}{3} + \dots = C + \sum_{n=0}^{\infty} c_n \frac{(x-\alpha)^{n+1}}{n+1}.$$

The radii of convergence of $f(x)$, $f'(x)$, and $\int f(x)dx$ are the same. But the intervals of convergence of $f(x)$, $f'(x)$, and $\int f(x)dx$ might be different as a result of the behavior at the endpoints.

Example 1: Radius of convergence equals 0

Given the power series $\sum_{n=0}^{\infty} n!x^n$. Determine the formula for the term a_n of the series and the coefficient c_n of the series. What is the center of the series?
Find the values of x for which the series converges. What is the radius of convergence of the series?

Solution

Write the solution here

Example 2: Radius of convergence equals ∞

Given the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$. Determine the formula for the term a_n of the series and the coefficient c_n of the series. What is the center of the series?
Find the values of x for which the series converges. What is the radius of convergence of the series?

Solution

Write the solution here

Example 3: Radius of convergence is a finite, positive number

Given the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x-3)^n$. Determine the formula for the term a_n of the series and the coefficient c_n of the series. What is the center of the series?
Find the values of x for which the series converges. What is the radius of convergence of the series?

Solution

Write the solution here

Example 4: Interval of convergence

Find the interval of convergence of the given series. Make sure to test the endpoints.

1. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$
2. $\sum_{n=2}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$

Solution

Write the solution here

Example 5: Term-by-term differentiation and integration

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \text{ for } x \text{ in the interval of convergence of the series.}$$

1. Find the series for $f'(x)$ and $\int f(x)dx$.
2. Find the interval of convergence of $f(x)$, $f'(x)$ and $\int f(x)dx$.

Solution

Write the solution here