Power Series

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A power series centered at α is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-\alpha)^n = c_0 + c_1(x-\alpha) + c_2(x-\alpha)^2 + \ldots$$

If the center $\alpha = 0$, the series becomes:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

E.g. $\displaystyle\sum_{n=0}^{\infty} \boxed{\underbrace{n!}_{c_n} x^n}$

Center $\alpha = 0$. nth Term: $a_n = n!\, x^n$

Coefficient $c_n = n!$

E.g. $\displaystyle\sum_{n=0}^{\infty} \underbrace{\boxed{\frac{(-1)^n}{2n+1}}}_{c_n} (x-3)^n$

Center $\alpha = 3$

nth coeff. $c_n = \dfrac{(-1)^n}{2n+1}$

nth term:

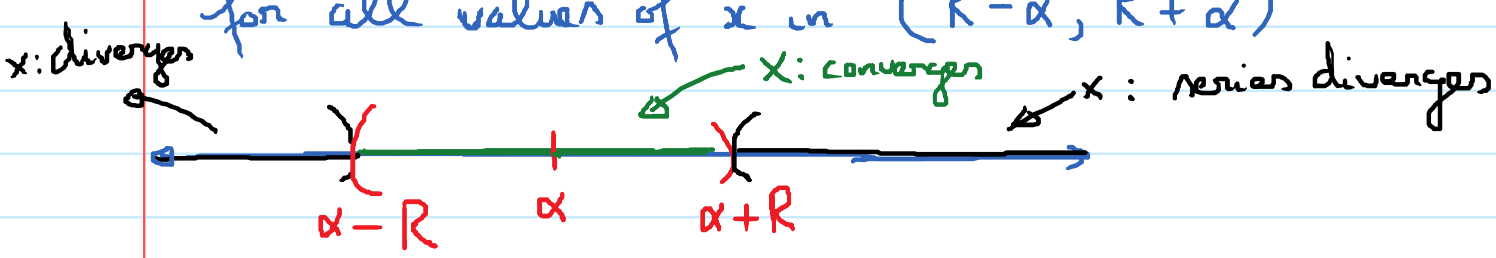
$a_n = \dfrac{(-1)^n}{2n+1}(x-3)^n$

let $\sum_{n=0}^{\infty} c_n (x-\alpha)^n$ be a power series.

There are exactly 3 possibilities:

1. The series converges at exactly one point $x = \alpha$
2. The series converges for all values of x.
3. The series converges for x values that belong to an interval surrounding the center.

There is a # $R > 0$ such that the series converges for all values of x in $(R-\alpha, R+\alpha)$



It diverges for x in $(-\infty, \alpha-R) \cup (\alpha+R, \infty)$

R: is called the radius of convergence of series

Note: Case 1: Radius of convergence $R = 0$

Case 2: Radius of convergence $R = \infty$

E.g. 1. $\sum_{n=0}^{\infty} n!\, x^n$

n^{th} term : $\boxed{a_n = n!\, x^n}$

n^{th} coeff: $c_n = n!$

center $\alpha = 0$

Apply ratio test to find values of x for which this series converges.

$$a_{n+1} = (n+1)!\, x^{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\cancel{!}\, x^{\cancel{n}+1}}{\cancel{n!}\, \cancel{x^n}}\right| = (n+1)|x|$$

$$\lim_{n\to\infty} (n+1)|x| = \begin{cases} 0 & \text{if } x = 0 \to \text{series converges} \\ \infty & \text{if } x \neq 0 \to \text{series diverges} \end{cases}$$

Conclusion: $\sum_{n=0}^{\infty} n!\, x^n$ converges for $x = 0$ only.

E.g. 2 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$

Center : $\alpha = 0$

n^{th} term : $a_n = \dfrac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$

n^{th} coeff : $c_n = \dfrac{(-1)^n}{2^{2n}(n!)^2}$

Ratio test:

$$a_{n+1} = \frac{(-1)^{n+1}}{2^{2n+2}[(n+1)!]^2} x^{2n+2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{x^{\cancel{2n}+2}}{2^{\cancel{2n}+2}[(n+1)!]^2} \cdot \frac{2^{\cancel{2n}}(n!)^2}{\cancel{x^{2n}}} \right|$$

$$= \left| \frac{x}{2(n+1)} \right|^2 = \frac{|x|^2}{4(n+1)^2}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^2}{4(n+1)^2} = 0 < 1$$

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The series converges regardless of what x is.

So, series converges for all values of x.

<u>E.g. 3</u> $$\sum_{n=0}^{\infty} \boxed{\frac{(-1)^n}{2n+1}(x-3)^n} \quad a_n$$

center $x = 3$.

$$a_{n+1} = \frac{(-1)^{n+1}}{2n+3}(x-3)^{n+1}$$

Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{\cancel{n}+1}}{2n+3} \cdot \frac{2n+1}{\cancel{(x-3)^n}}\right|$$

$$= \frac{2n+1}{2n+3}|x-3|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n+1}{2n+3}|x-3| = |x-3|$$

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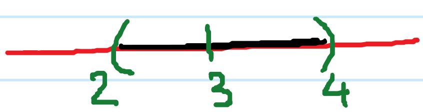
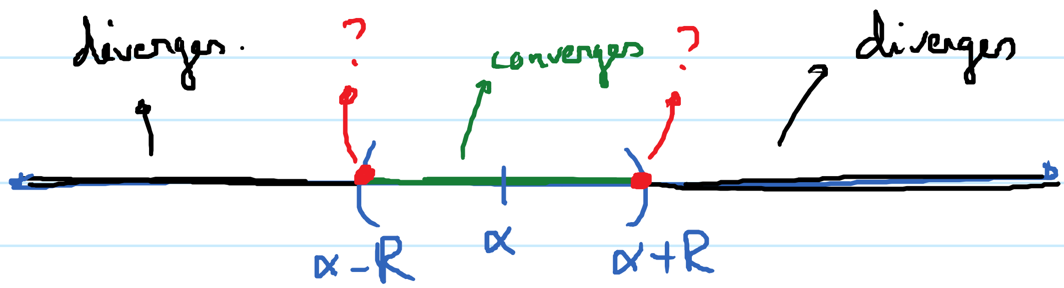
For series to converges, we want $|x-3| < 1$.



Radius of convergence $= 1$

Of the 3 scenarios that we've seen:

1. Interval of convergence $= \{\alpha\}$
2. Interval of convergence $= (-\infty, \infty)$
3. Interval of convergence will contain the interval $(R-\alpha, R+\alpha)$



The result did not say any-thing about convergence or divergence at the end point $x = \alpha - R$ or $x = \alpha + R$.

Hence, to determine the precise interval of convergence we need to test these endpoints by plugging $x = \alpha - R$ and $x = \alpha + R$ back to the original series and apply tests.

In this case, the possibilites of I.O.C. are:

$(\alpha - R, \alpha + R)$; $[\alpha - R, \alpha + R)$; $(\alpha - R, \alpha + R]$; $[\alpha - R, \alpha + R]$

E.g. 4. Find I.O.C. of the given series:

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}.$$

$$a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{\cancel{n}+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{\cancel{(x-2)^n}}\right|$$

$$= \frac{n^2+1}{(n+1)^2+1} \cdot |x-2|$$

$$\lim_{n\to\infty} \underbrace{\frac{n^2+1}{(n+1)^2+1}}_{1} |x-2| = |x-2|$$

For series to converge: $|x-2| < 1$

Radius of convergence