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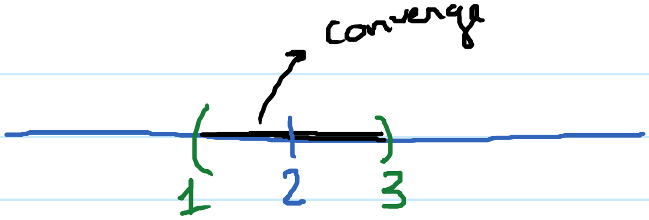


Test for convergence at endpoints : $x=1$ and $x=3$

$\underline{x=1:}$ $$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

Try A.S.T $a_n = \dfrac{1}{n^2+1}$

① $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{n^2+1} = 0$ ✓

② a_n is nonincreasing $a_{n+1} \le a_n$. ✓

$$\frac{1}{(n+1)^2+1} \le \frac{1}{n^2+1}$$

By A.S.T, the series converges.

$x=3:$ $$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

$$n^2+1 > n^2 \to \frac{1}{n^2+1} < \frac{1}{n^2} \to \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges

p-series

$p=2$

The series $\sum \frac{1}{n^2+1}$ converges (b/c it is < a convergence series)

Conclusion: I.O.C : $[1, 3]$

② $$\sum_{n=2}^{\infty} \frac{n!\, x^n}{1\cdot 3\cdot 5 \cdots (2n-1)} \to a_n$$

$2(n+1) - 1$

$$a_{n+1} = \frac{(n+1)!\, x^{n+1}}{1\cdot 3\cdot 5 \cdots (2n-1)\underline{(2n+1)}}$$

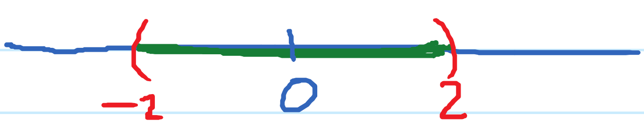
$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(n+1)\cancel{!}\, x^{\cancel{n+1}}}{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdots(2\cancel{n-1})(2n+1)} \cdot \frac{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdots(\cancel{2n-1})}{\cancel{n!}\, \cancel{x^n}} \right|$$

$$= \frac{n+1}{2n+1}|x|$$

R.O.C

$$\lim_{n\to\infty} \frac{n+1}{2n+1}|x| = \frac{1}{2}|x|$$

For series to converge, $\frac{1}{2}|x| < 1 \to |x| < 2$



Test endpoints:

$$x=2: \quad \sum_{n=2}^{\infty} \frac{n!\, 2^n}{1\cdot 3\cdot 5\cdots(2n-1)}$$

$$= \sum_{n=2}^{\infty} \frac{(1\cdot 2\cdot 3\cdots n)\overbrace{(2\cdot 2\cdot 2\cdots 2)}^{n \text{ times}}}{1\cdot 3\cdot 5\cdots(2n-1)}$$

$$= \sum_{n=2}^{\infty} \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}$$

$$\lim_{n\to\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} \neq 0 \quad (\text{b/c num.} > \text{deno.})$$

So, series diverges b/c of nth term test

$$x=-2: \quad \sum_{n=2}^{\infty} \frac{n!\,(-2)^n}{1\cdot 3\cdot 5\cdots(2n-1)}$$

series diverges b/c of n^{th} term test.

I.O.C: $\boxed{(-2, 2)}$

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$$f(x) = \sum_{n=0}^{\infty} \boxed{c_n (x-\alpha)^n} = \underbrace{c_0 + c_1(x-\alpha) + c_2(x-\alpha)^2 + c_3(x-\alpha)^3 + \cdots}$$

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-\alpha)^{n-1} = c_1 + 2c_2(x-\alpha) + 3c_3(x-\alpha)^2 + \cdots$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-\alpha)^{n+1}}{n+1} + C = C + c_0 x + c_1 \frac{(x-\alpha)^2}{2} + c_2 \frac{(x-\alpha)^3}{3} + c_3 \frac{(x-\alpha)^4}{4} + \cdots$$

Note: The Radii of convergence of $f'(x)$ and $\int f(x)\,dx$ are the same as that of $f(x)$. But the interval of convergence might be different because of the behavior at the endpoints.

Eg. 5. $f(x) = \sum_{n=1}^{\infty} \boxed{\frac{x^n}{n}}$.

(1) Find the series for $f'(x)$ and $\int f(x)\,dx$

$$f'(x) = \sum_{n=2}^{\infty} \frac{\cancel{n} x^{n-1}}{\cancel{n}} = \sum_{n=2}^{\infty} x^{n-1}.$$

$$\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} + C.$$

(2) I.O.C. of the series for $f(x)$, $f'(x)$ and $\int f(x)dx$

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{x^n}}\right| = \frac{n}{n+1}|x|$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n}{n+1}|x| = |x|$$

$\to 1$

For series to converge, $|x| < 1$.

I.O.C. contains $(-1, 1)$

Check endpoints: $x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series $p = 1$)

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges b/c A.S.T.

Conclusion: I.O.C. for $f(x) = [-1, 1)$

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$$f'(x) = \sum_{n=2}^{\infty} x^{n-1}$$

I.O.C. contains $(-1,1)$

- $x = 1$: $\sum_{n=2}^{\infty} (1)^{n-1} = \sum_{n=2}^{\infty} 1$ diverges

(b/c n^{th} term test)

- $x = -1$: $\sum_{n=2}^{\infty} (-1)^{n-1}$ diverges (b/c of n^{th} term test)

I.O.C. for $f'(x)$: $(-1,1)$

$$\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$$

I.O.C. contain $(-1,1)$.

$\underline{x=1:}$ $C + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ direct comparing this with $\sum \frac{1}{n^2}$

$$n(n+1) > n^2 \rightarrow \frac{1}{n(n+1)} < \frac{1}{n^2}.$$

$$\sum \frac{1}{n(n+1)} < \sum \frac{1}{n^2}$$

$\rightarrow$ converges b/c p-series ; $p = 2$.

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$x = -1$: $C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}$ converges by A.S.T.

Conclusion: I.O.C. for $\int f(x)dx$: $[-1, 1]$