Representation of Functions by Power Series.

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Geometric power series:

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

mult. by x mult. by x mult. by x

→ Geometric power series. Common Ratio $R = x$

→ It converges if $|x| < 1$. I.O.C: $(-1, 1)$

→ $\dfrac{1}{1-x}$

We say that the function $f(x) = \dfrac{1}{1-x}$ on $(-1,1)$ is represented by the series $1 + x + x^2 + x^3 + x^4 + \cdots$

$$\boxed{f(x) = \boxed{\frac{1}{1-x}} = \boxed{\sum_{n=0}^{\infty} x^n} \;;\; \text{I.O.C. } \boxed{(-1,1)}}$$

Function Series centered at 0 I.O.C

$1 - x + x^2 - x^3 + x^4 - x^5 + \dots$

$\rightarrow$ Geometric power series centered at 0.

Common Ratio : $R = -x$.

I.O.C. $|-x| < 1 \rightarrow |x| < 1$

$\rightarrow (-1, 1)$

converges to : $\dfrac{1}{1+x}$.

$$\boxed{f(x) = \frac{1}{1+x} = \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}\ ;\ \text{I.O.C. } (-1,1)}$$

Function

Series centered at 0

Eg.1 Functions that can be represented by geometric series:

(1) $f(x) = \dfrac{1}{x+2}$. Find a geometric power series centered at 0 for f.

Note: $\boxed{\dfrac{1}{1-R} = \sum_{n=0}^{\infty} R^n}$. (geometric)

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{1+\frac{x}{2}}$$

$$= \frac{1}{2} \cdot \boxed{\frac{1}{1-\left(-\frac{x}{2}\right)}} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

$R \longrightarrow$ common ratio

$$f(x) = \frac{1}{x+2} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{2^n} .$$

$$|R| = \left|-\frac{x}{2}\right| < 1 \rightarrow \left|\frac{x}{2}\right| < 1$$

$$\rightarrow |x| < 2 \rightarrow \text{I.O.C. } \boxed{(-2, 2)}$$

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(2) $f(x) = \frac{2}{6-x}$

Find a geo. power series for f centered at -2.

$$\frac{2}{8-(x+2)} = \frac{2}{8\left[1-\left(\frac{x+2}{8}\right)\right]}$$

$$= \frac{2}{8} \cdot \frac{1}{1-\frac{x+2}{8}} = \frac{1}{4} \cdot \frac{1}{1-\underbrace{\frac{x+2}{8}}_{R}}$$

$$= \frac{1}{4} \cdot \sum_{n=0}^{\infty} \left(\frac{x+2}{8}\right)^n$$

$$\boxed{f(x) = \frac{2}{6-x} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(x+2)^n}{8^n}}$$

I.O.C. $|R| < 1 \rightarrow \left|\frac{x+2}{8}\right| < 1$

$|x+2| < 8 \rightarrow -8 < x+2 < 8$

$\rightarrow -10 < x < 6$. I.O.C. $\boxed{(-10,6)}$

Basic operations on power series.

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$; $g(x) = \sum_{n=0}^{\infty} d_n x^n$

Then:

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} c_n x^n \pm \sum_{n=0}^{\infty} d_n x^n$$

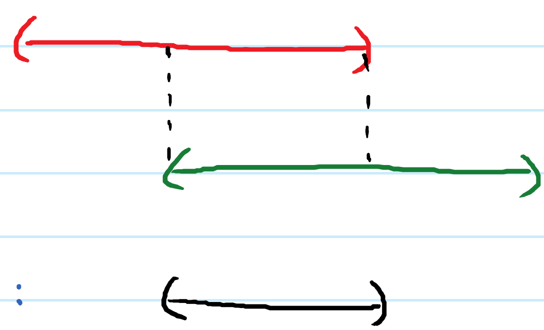
$$= \sum_{n=0}^{\infty} c_n x^n \pm d_n x^n$$

$$= \sum_{n=0}^{\infty} (c_n \pm d_n) x^n$$

Note: I.O.C. of f: I_1

I.O.C of g: I_2

I.O.C of $f \pm g$: $I_1 \cap I_2$:



* $f(x) = \sum_{n=0}^{\infty} c_n x^n$

$$f(x^p) = \sum_{n=0}^{\infty} c_n (x^p)^n = \sum_{n=0}^{\infty} c_n x^{pn}$$

I.O.C. of $f(x)$: I.

I.O.C of $f(x^p)$ can be obtained by solving for x so that x^p is in I.

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$f(kx) = \sum_{n=0}^{\infty} c_n (kx)^n = \sum_{n=0}^{\infty} c_n k^n x^n .$$

I.O.C. of $f(x)$ is I. $(-1, 5)$

I.O.C. of $f(kx)$ in x so that kx is in I.

Diff. and Int. power series for f to obtain power series for $f'(x)$; $\int f(x)\,dx$.

E.g. 2

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{I.O.C. } (-1,1)$$

$$u(x) = \frac{1}{1-4x^2} = f(4x^2) = \sum_{n=0}^{\infty} (4x^2)^n$$

$$\boxed{u(x) = \sum_{n=0}^{\infty} 4^n x^{2n}}$$

I.O.C. : $-1 < 4x^2 < 1 \rightarrow -\frac{1}{4} < x^2 < \frac{1}{4}$

$$\boxed{-\frac{1}{2} < x < \frac{1}{2}}$$

2) $$v(x) = \frac{x}{2x^2+1}$$

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n . \quad \text{I.O.C. } (-1, 1)$$

$$v(x) = x \cdot \frac{1}{1+2x^2} = x \cdot f((2x^2))$$

$$= x \cdot \sum_{n=0}^{\infty} (-1)^n (2x^2)^n$$

$$= x \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n}$$

$$\boxed{v(x) = \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n+1} .}$$

I.O.C. $\quad -1 < 2x^2 < 1$

$$-\frac{1}{2} < x^2 < \frac{1}{2}$$

$$\boxed{-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}}$$

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E.g. 3:

$$f(x) = \frac{3}{x^2 - x - 2} = \frac{3}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$3 = A(x-2) + B(x+1)$$

Plug in $x = 2$: $3 = 3B \rightarrow B = 1$

$x = -1$; $3 = -3A \rightarrow A = -1$

$$\boxed{\frac{3}{x^2 - x - 2}} = \boxed{\frac{-1}{x+1}} + \boxed{\frac{1}{x-2}}$$

$$\frac{-1}{x+1} = -1 \cdot \frac{1}{1+x} = -\sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} x^n$$ I.O.C. $(-1, 1)$

$$\frac{1}{x-2} = \frac{1}{2-x} = \frac{1}{2\left(1 - \frac{x}{2}\right)}$$

$-1 < \frac{x}{2} < 1$

$-2 < x < 2$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{x}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$ I.O.C. $(-2, 2)$